COS 522 Complexity — Homework 1.

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Total of 110 points. Due March 6th, 2006.
(There is no class on March 6th, so please email or put it in my mail box on March 6th or latest by March 7th morning.)

Exercise 1 (22 points)**.** Prove that $\mathbf{SPACE}(n) \neq \mathbf{NP}$. (Note that as far as we know it may be that $\mathbf{SPACE}(n) \subseteq \mathbf{NP}$ or that $\mathbf{NP} \subseteq \mathbf{SPACE}(n)$.

Exercise 2 (22 points)**.** Consider the field $\mathbb{F} = \mathsf{GF}(2)$ (computation modulo 2), and the problem of multiplying two $n \times n$ matrices A and B over this field. While both the input and output can be described in $O(n^2)$ bits, it is not known for this field (or any other field) whether there is an $O(n^2)$ time algorithm to compute this product.[1] However, *verifying* a solution is easier:

Give a $O(n^2)$-time probabilistic algorithm V such that for every three $n \times n$ matrices A, B, C, if $AB = C$ then $\Pr[V(A,B,C) = 1] = 1$ and if $AB \neq C$ then $\Pr[V(A,B,C) = 1] \leq 1/2$.

Exercise 3 (22 points)**.** Show that that the random walk idea does not work for solving $s-t$ connectivity on *directed* graphs. That is, show a directed graph G with n vertices and no sinks such that there is a path from s to t, but the probability that a random walk from s reaches t within $2^{n^{1/10}}$ steps is less than $2^{-n^{1/10}}$ (this can be improved to $2^{\Omega(n)}$ steps and probability $2^{-\Omega(n)}$).

Exercise 4 (22 points)**.** Prove the Valiant-Vazirani Lemma (Lemma 8.9, page 117): let $S \subseteq \mathsf{GF}(2)^n$ be a nonempty set. Then, there exists $i \in \{1, \ldots, n\}$ such that if we choose $a^1, \ldots, a^i \leftarrow_{\mathrm{R}} \mathsf{GF}(2)^n$ to be i independent n-dimensional vectors over $\mathsf{GF}(2)$, and define the function $h : \mathsf{GF}(2)^n \to \mathsf{GF}(2)^i$ as follows $h(s)_k = \sum_{j=1}^{n} s_j a^k_j \pmod 2$ then

$$\Pr\Big[\big|\{s \in S \mid h(s) = 0^i\}\big| = 1\Big] \geq \frac{1}{10}$$

Exercise 5 (22 points)**.** Let $G = G(V,E)$ be an n vertex d-regular undirected graph (for some $d \geq 3$) with normalized adjacency matrix A having second largest eigenvalue at most $1/2$. Prove that for every set $S \subseteq V$ with $|S| < n/20$, it holds that $|\Gamma(S) \setminus S| \geq |S|/10$. (We denote by $\Gamma(S)$ the set of neighbors of S, that is the set of vertices v such that $(u,v) \in E$ for some $u \in S$.)

[1] The natural algorithm takes $O(n^3)$ time, but there are faster algorithms (taking roughly $O(n^{2.4})$ steps) and the exact complexity of the problem is not known.