Multiple-Object Auctions with Budget Constrained Bidders*

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Abstract

A seller with two objects faces a group of bidders who are subject to budget constraints. The objects have common values to all bidders, but need not be identical and may be either complements or substitutes. In a simple complete information setting we show: (1) if the objects are sold by means of a sequence of open ascending auctions, then it is always optimal to sell the more valuable object first; (2) the sequential auction yields more revenue than the simultaneous ascending auction used recently by the FCC if the discrepancy in the values is large, or if there are significant complementarities; (3) a hybrid simultaneous-sequential form is revenue superior to the sequential auction; and (4) budget constraints arise endogenously.

1 Introduction

In the last few years governments in many parts of the world have aggressively sought to privatize socially held assets. This wave of privatization has included the sale of industrial enterprises in the former Soviet Union and Eastern Europe, the sale of public transportation systems in Britain and Scandinavia and the sale of radio spectra in the US. Both the enormity and the obvious importance of these sales have naturally led to an examination of how best to accomplish this task. Auctions have been proposed, and have frequently been used, as time-honored, convenient and attractive mechanisms for such sales. This, in turn, has led to a revival of interest in the theory of auctions, particularly those involving multiple objects.

The magnitude of these privatization sales has meant that in many instances it is realistic to consider that buyers may run up against liquidity or borrowing constraints. The

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presence of these financial constraints introduces important differences into traditional auction theory. For instance, Pitchik and Schotter (1986) and Che and Gale (1996, 1998) have shown that when bidders are subject to such constraints, the conclusion of the celebrated revenue equivalence theorem no longer holds.

When multiple objects are auctioned in the presence of budget constraints, it may be advantageous for a bidder to bid aggressively on one object with a view to raising the price paid by his rival and depleting his budget so that the second object may then be obtained at a lower price. In effect, a bidder may wish to "raise a rival's costs" in one market in order to gain advantage in another. Thus, unlike in traditional auction theory, a particular bidder's payoff is affected by the price paid by a rival bidder.[1] It is important to note that this effect comes into play only when several objects are sold.

Budget constraints seem to have played a significant role in the auctions for radio spectrum licenses conducted recently by the Federal Communications Commission (FCC) (see McMillan (1994)). In particular, Salant (1997) writes that the assessment of rival bidders' budget constraints was a primary component of the pre-bidding preparation of GTE's bidding team. Moreover, confirming that the kinds of considerations mentioned above played a significant role, he describes the strategic advantages arising from bidding for licenses of secondary importance so as to "make rivals spend more on some markets, leaving them with less to spend in other markets," (Salant (1997), p. 561).

In this paper we study a simple model in which two objects are sold to a group of financially constrained bidders.[2] Our model is one of *complete* information: the values of the objects and the budgets facing the bidders are all commonly known to all participants. Without budget constraints, this setting would be too trivial to be of any interest. With budget constraints, however, this "bare bones" structure turns out to be sufficiently rich to capture and highlight a number of interesting issues. The model is very close to that studied by Pitchik and Schotter (1988). Their experiments confirm that the strategic considerations introduced by budget constraints play an important role in practice, as well as in theory.

Sequential Auctions and the Order of Sale Suppose that two precious paintings are being auctioned sequentially, say in two successive English auctions. In a complete information setting without budget constraints, whether painting A, the more valuable one, is sold before or after painting B does not affect the price that either fetches. ¿From the perspective of the seller, the order of sale is irrelevant. However, when buyers are financially constrained the order of sale can affect the total revenue accruing to the seller. We show below that if object A is more valuable than object B, the revenue derived from selling them sequentially in the order AB is at least as large as the revenue from selling them in the order BA (Proposition 1 below).

[1] This strategic linkage across goods is emphasized by Pitchik and Schotter (1986, 1988).

[2] In the first part of the paper we suppose that the constraints faced by the bidders are exogenously given, possibly the result of liquidity or credit constraints. Later, we show that budget constraints may arise endogenously .

Simultaneous Auctions An alternative auction design is to sell the objects by means of two simultaneous auctions. This was the design favored by the FCC in the spectrum auctions mentioned above, but to the best of our knowledge its theoretical properties have not been studied. In our simple setting we are able to compare the revenue obtained from the simultaneous auction with the optimal sequential auction when the objects are complements. We find that the optimal sequential auction outperforms the simultaneous auction in terms of revenue when (a) the values of the two objects are substantially different; or when (b) there are significant complementarities (Proposition 2).

We then introduce a hybrid auction form that combines essential elements from the simultaneous and sequential forms. We show that this hybrid form is revenue superior to the sequential form (Proposition 3).

Endogenous Budgets The FCC auction also serves to introduce the second main theme of this paper: that budget constraints may arise *endogenously* as a result of rational calculation on the part of bidders.

Some of the larger bidders in the FCC auction consisted of consortia of large telecommunication companies: WirelessCo (a consortium including Sprint and the three largest cable companies) and PCS PrimeCo (a consortium including NYNEX, Bell Atlantic and US West). Other participants, bidding as individual companies (for example, GTE, Bell South and PacTel), were themselves rather large local exchange carriers. In what sense were these companies financially constrained? While some of the smaller companies could be said to face constraints on borrowing or liquidity, this is less clear for these larger bidders. Indeed, Salant (1997, p. 553) writes that the bidding team first assessed the values of licenses that GTE was eligible to bid on and the "budget parameters *determined* by GTE's management were in part based on these valuations." [Emphasis added.] Thus it seems that for the larger bidders the budget constraints were endogenously determined. The companies gave their bidding teams instructions not to spend more than a specified amount.

We study below a simple model in which budget constraints are endogenously determined prior to the sale of the objects. In other words, the choice of a budget is itself a strategic decision. Our main result is that budget constraints *always* arise : the resulting game has an (essentially) unique equilibrium that involves constraints that bind (Proposition 4). Thus our model provides an explanation of how budget constrained bidding may be the result of a conscious choice rather than the result of exogenous factors like liquidity constraints or capital market imperfections.

Miscellaneous Results When multiple objects are sold, budget constraints can have some unanticipated consequences. For example, a reserve price can raise the seller's revenue even though it is set at such a low level that it is never binding in equilibrium. Similarly, there may be a trade-off between efficiency and revenue even though the setting is one of complete information. In a separate section, we illustrate some of these phenomena by means of examples.

Related Literature An early discussion of some issues concerning budget constraints and auctions is contained in Rothkopf (1977). His analysis is not game-theoretic, rather it focuses on how the computation of best responses is affected by budget constraints.

Palfrey (1980) has studied the effects of budget constraints in a multiple object setting with complete information. Specifically, he analyzes sealed-bid discriminatory ("pay as you bid") auctions and characterizes equilibrium in the two object, two bidder case. He also points out difficulties when the number of objects or bidders is larger. In particular, equilibrium may not exist and when it does, may not be unique.

Engelbrecht-Wiggans (1987) studies the different forms that financial constraints may take. Bidders may be constrained to spend no more than a certain amount absolutely. Alternatively, they may be constrained to spend no more than a certain amount on average. Engelbrecht-Wiggans (1987) studies how these different forms affect the range of admissible strategies and derives some equivalence relations.

Our work is closely related to two papers by Pitchik and Schotter (1986, 1988), which also analyze sequential auctions with budget-constrained bidders. The basic setting and modes of reasoning are the same. However, we extend their model and analysis in many directions. We allow for more than two bidders and the possibility of synergies between the objects.[3] We also consider simultaneous open auctions. Another important difference is that in our model (in Section 5 below) budgets may be determined endogenously. As will become clear, our goals and the questions we ask are also quite different. Pitchik and Schotter (1988), in particular, is focused towards deriving testable predictions of equilibrium bidding behavior which can then be measured against data from experiments.

Pitchik (1995) studies two-bidder sequential auctions of two objects when there is incomplete information. Bidder types determine both valuations and budgets. She also studies how the sequence of sale affects both the total revenue and the prices of the individual objects. We discuss the relationship of her results to ours in Sections 3 and 7 below.

Che and Gale (1996 and 1998) study single object auctions with budget constraints under incomplete information. They show that even in the traditional setting with independent private values and symmetric bidders, revenue equivalence between various auction formats no longer holds. The second paper, Che and Gale (1998), provides conditions under which the first price auction outperforms a second price auction in terms of revenue. They also show how the use of lotteries and all-pay auctions can benefit the seller when bidders are budget constraint.

Lewis and Sappington (1998) consider agency problems when agents have budget constraints, again showing that contracts that incorporate all-pay features may be advantageous.

We emphasize that in the existing literature budget constraints are always taken to be given exogenously.

The results described in the introduction are presented below. We have relegated all

[3]These extensions are not without consequences. In particular, as Pitchik and Schotter (1986) themselves anticipate, some of their results are affected by the presence of additional bidders. See Section 3 below for details.

formal proofs to an appendix.

2 The Model and an Example

There are n bidders with budgets $y_1 \geq y_2 \geq y_3 \geq \ldots \geq y_n$ and two objects, A and B. The value derived from obtaining A alone is V^A, while the value of B alone is V^B, where $V^A \geq V^B > 0$. The value of the two objects as a bundle is denoted by V^{AB}, which may be smaller or larger than $V^A + V^B$. These known values are common to the bidders. It is convenient to write:

$$V^{AB} = V^A + V^B + \alpha$$

where α is the *synergy,* and this may be positive or negative or zero. When the synergy is positive we say that the objects are *complements*; when the synergy is negative the objects are *substitutes*. We assume that $\alpha > -V^B$, so that the marginal value of either object is always positive.

An Example We begin with a simple example which illustrates the sorts of considerations that arise in multiple object auctions with financially constrained bidders. The example also shows that budget constraints may have an important impact even when these constraints are relatively liberal.

Suppose that the objects are sold sequentially by means of two successive English auctions and that there are three bidders with budgets y_1, y_2 and y_3, respectively. There are no synergies ($\alpha = 0$), so the value of obtaining both objects is simply the sum of the two values. In the absence of any budget constraints each object would, of course, fetch its full value in an English auction and the seller would receive a total revenue of $V^A + V^B$.

Now consider the following parameter values:

Example 1 *Values:* $V^A = 50, V^B = 40, V^{AB} = 90$. *Budgets:* $y_1 = 100, y_2 = 80, y_3 = 20$.

Observe that the budget constraints are fairly weak. In particular, bidder 1 is effectively unconstrained since $y_1 > V^A + V^B$, so that 1 can afford both objects at prices that equal their respective values. Bidder 2 is constrained but $y_2 > V^A > V^B$, and thus 2 can afford to buy either object at a price equal to its value. Suppose that the objects are sold in the order A followed by B.

In the second auction it is a weakly dominant strategy for each bidder to bid the minimum of his remaining budget and V^B. If bidder 1 wins object A in the first auction, then in the second auction both bidders 1 and 2 will have residual budgets that exceed V^B. Thus, B will then sell for $V^B = 40$ and the second auction will yield no surplus. Therefore, it is dominated for bidder 2 to drop out of the bidding for A before the price reaches 50. If bidder 1 wins A for 50, his net total gain will be 0. Bidder 1 is better off letting 2 win the first object for (just below) 50 and then winning the second object for (just above) $(80-50) = 30$. The total revenue to the seller is 80 and the full value of the objects is not realized.

Equilibrium concept. A word on the equilibrium concept that we use is in order. In the second auction, we suppose that no bidder plays a weakly dominated strategy. Thus, in every subgame each player bids up to the minimum of the value of the object to him and his current budget. These outcomes are taken as given in the first auction, and we look for an equilibrium in undominated strategies in the resulting reduced game.

For formal reasons, it is necessary to assume that there is a smallest unit of currency in which bids are made, say, one cent, although we do not keep precise track and in our reasoning equate, say, \$5.99 with \$6. See Pitchik and Schotter (1986, 1988) for a similar treatment.

3 Sequential Open Auctions and the Order of Sale

In this section we ask which order of sale is advantageous from the perspective of the seller. It is instructive to begin with an example.

Example 2 *Values:* $V^A = 50, V^B = 39, V^{AB} = 89$. *Budgets:* $y_1 = 55, y_2 = 30,$ *and* $y_3 = 20$.

Since $y_3 = 20$, both objects sell for at least 20, so that, in particular, object B cannot yield a surplus greater than 19. Thus, when the objects are sold in the order AB, bidder 1 would be willing to bid up to $p_1^A = 31$ in the first auction whereas bidder 2's budget is only 30. Bidder 1 wins A for 30, and then bidder 2 wins B for $y_1 - 30 = 25$. Total revenue in the order AB is $R^{AB} = 55$.

In the order BA, bidder 2 must bid up to $p_2^B = 25$ in the first auction because if he drops out before that bidder 1 will win both objects. At a price of 25, bidder 1 will drop out of the bidding for good B and then go on to win good A in the second auction for a price of 20. Total revenue in the order BA is $R^{BA} = 45$, and $R^{BA} < R^{AB}$.

Notice that in both orders bidder 1 wins A and bidder 2 wins B, and so the allocation is the same. What matters, however, is that bidder 1 wins the *first* good in the order AB, whereas he wins the *second* good in the order BA. The seller is generally better off when 1 wins the first good. To see this, consider the simple two bidder case: $i)$ $V^A > V^B > y_1 > y_2 > 0$ and $ii)$ $y_1 < 2y_2$. Assumption $i)$ implies that in the second auction a bidder will be willing to bid up to his budget constraint, while $ii)$ implies that bidder 1 does not have enough money to win both goods and so each bidder wins one good. Suppose bidder i wins the first good for a price p. The second good will then sell for $y_i - p$, so that total revenue will be $p + y_i - p = y_i$. Since $y_1 > y_2$ the seller is better off if bidder 1 wins the first good. Since, as one would expect, bidder 1 is more prone to win the first good in the order AB than in the order BA, the order AB is better.

In the previous example each bidder wins exactly one object in each order since $y_1 < 2y_2$. In the next example bidder 1 has enough money to outbid 2 on both objects.

Example 3 *Values:* $V^A = 60, V^B = 40, V^{AB} = 112$. *Budgets:* $y_1 = 70, y_2 = 30, y_3 = 5$.

In the order *AB* bidder 1 wins both objects, each for 30, and gains a net surplus of 52. The total revenue is $R^{AB} = 60$. In the order *BA* bidder 1 is willing to win *B* for a price up to 27. At that price, if 1 wins *B* he will then go on to win *A* for 30, yielding him a total surplus of 55. On the other hand, if he drops out at 27 he will win *A* for a price of 5, once again resulting in a surplus of 55. Since bidder 2 will lose both auctions if he loses the first, 2 is willing to bid up to 30 on *B*. However, 1 has no incentive to push 2 beyond a price of 27 since, in any case, he will not win the second object *A* for less than $y_3 = 5$.Thus bidder 1 will drop out at 27. Bidder wins *A* for 27 and Bidder 1 goes on to win *A* for 5. The total revenue is now $R^{BA} = 32$. Once again, the revenue from the order *AB* is higher.

In this example bidder 1 wins both objects in the order *AB* and only *A* in the order *BA*. When he wins both objects each is sold for a price of y_2. When he wins only one, the price of at least one of the objects is lower than y_2. Thus the seller is better off when bidder 1 wins both objects. Again, it can be argued that bidder 1 is more prone to win both when the order is *AB* rather than *BA*.

Our first result shows that Examples 2 and 3 are instances of a general phenomenon.[4]

Proposition 1 *The revenue to the seller from selling the objects sequentially in the order AB is at least as great as the revenue from selling them in the order BA. For some parameter values, the inequality is strict.*

In a related incomplete information setting, Pitchik (1995) also examines how the seller's revenue is affected by the order of sale. In her model, each bidder is identified by a one-dimensional "type" and both the valuations and the budgets are affine functions of a bidder's type. In addition, budgets are assumed to be at "intermediate" levels so that in equilibrium each bidder wins exactly one object for all realizations of the types. Pitchik (1995) then shows that the optimal order of sale depends on whether the "de facto valuation" for the first object is an increasing or decreasing function of the type. Her results are not directly comparable to Proposition 1 because hers is a private value setting and it is not the case that a particular object, say *A*, is considered more valuable by all types.

More closely related is a result of Pitchik and Schotter (1986) and it is worthwhile to compare this to Proposition 1. Let p^A and q^B be the prices of *A* and *B*, respectively, when the order of sale is *AB*. Similarly, let p^B and q^A be the prices in the order *BA*. According to Proposition 1, $p^A + q^B \geq p^B + q^A$. Pitchik and Schotter (1986) show that when there are only two bidders, the price of an object is higher the earlier it is sold in the sequence, that is, $p^A \geq q^A$ and $p^B \geq q^B$. This result, however, does not hold generally when there are three or more bidders.[5]

What factors determine the order of sale in real-world auctions? There is some "psychological" intuition that suggests that it may be advantageous to "warm up the room" with some lower valued objects before bringing the auction to a climax with the more valuable

[4] For degenerate parameter values there may be more than one equilibrium revenue in a given order (for instance, $y_1 = 30, y_2 = 10, V^A = 20, V^B = 10$ with the order *AB*). We ignore these cases.

[5] Suppose $V^A = 100$, $V^B = 18$; and budgets are $y_1 = 50$, $y_2 = 20$, $y_3 = 4$. In the order *AB*, prices are $p^A = 20$ and $q^B = 18$. In the order *BA*, prices are $p^B = 16$ and $q^A = 4$. Thus $p^B < q^B$.

masterpiece. But psychology also suggests that it may be preferable to sell more valuable items first in order to establish "lively" bidding for the rest of the items (see Cassady (1967, pp. 84-85) for a discussion of these tactics.) Budget constraints, on the other hand, introduce pure economic considerations of the kind identified in this paper. Auction houses tend to balance these various effects.[6] As a concrete example, in an auction of eight paintings by Paul Cézanne conducted by the auction house of Sotheby's in November 1997, the prices of the paintings, in order of sale, were as follows:

CÉZANNE AUCTION

Painting	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Price (in $ millions)	$0.3	$1.0	$5.5	$1.1	$3.6	$5.9	$1.0	$0.7

Assuming that more valuable paintings go for higher prices, this does not reveal an overall monotonic pattern, although the values do decline at the end.[7] This is not atypical of the results from sales at Sotheby's (see Cassady (1967, pp. 295-298), for other examples).

4 Simultaneous Ascending Auction

In the recent auction of radio spectrum licenses the FCC adopted a novel auction design based on the recommendation of a group of economists. The licenses were sold using a *simultaneous ascending auction*. As in an English auction, there were multiple rounds of bidding that continued until no bidder wished to raise the bid. All auctions were conducted simultaneously and bidding on all objects remained open as long as there was bidding activity on any one of them.

A primary reason for adopting this design was the presence of positive synergies between licenses in adjoining areas. While combination bids, that is, bids on bundles of objects were not allowed, it was felt that the simultaneous nature of the auction would allow bidders to assemble bundles that exploited the synergies in an efficient manner.[8]

In this section we study a simple version of the simultaneous ascending auction when budget constraints are present and there are positive synergies between the objects ($\alpha \geq 0$). Specifically, we consider a simultaneous auction with the following rules:

1. There are multiple rounds of bidding conducted in the open.

[6]In private communication (Sotheby's (1998)) a representative of Sotheby's related to us various factors that they consider in determining the order of sale. In particular, budget issues were explicitly mentioned. Both of the behavioral concerns mentioned above also played a role. In some instances, there was a natural historical order associated with the items (paintings from an early period versus a later period) which dominated all other considerations.

[7]Proposition 1 concerns two object auctions. Section 6 below contains an example (Example 11) of an auction with more than two objects.

[8]See McMillan et al. (1997) and Cramton (1997) for both a description and the rationale underlying the choice of the auction rules. Ausubel et al. (1997) use bidding data from the auction to confirm the presence of positive geographic synergies.

2. In each round for *every* object, a bidder either

   (a) raises the previous high bid on the object; or

   (b) does not announce a bid on the object.

3. At no time can the total of a bidder's outstanding high bids exceed his budget.

4. The auction continues until, in two successive rounds *no* object has had its bid raised.

Note that with these rules, a bidder who has been outbid on, say, object A can first pursue the bidding on B before deciding whether to continue further bidding on A. Allowing for this possibility was an important feature of the FCC auction.[9]

In the simultaneous auction, we look for Nash equilibria once (weakly) dominated strategies have been *iteratively* removed.

In a simple two-object setting with complementarities and budget constraints, we compare the performance, in terms of both revenue and efficiency, of the simultaneous format outlined above with that of the sequential format of the previous section. While the equilibrium of the sequential auction is almost always unique, it is not surprising that the simultaneous auction often has multiple equilibria. Nevertheless, in some circumstances a ranking of the auction methods is possible.

We begin with an example.

Example 4 *Values:* $V^A = 20$, $V^B = 10$, $V^{AB} = 30$. *Budgets:* $y_1 = 25$, $y_2 = 6$, $y_3 = 1$.

In the sequential AB auction bidder 1 wins both A and B, each at a price of 6. The total revenue from the sequential auction $R^{AB} = 12$.

One equilibrium of the sequential auction has the following bidding behavior (where $\left(b_i^A, b_i^B\right)$ denotes the bids of bidder i on A and B):

Round I: 1 bids $(1.01, 5.99)$; 2 bids $(0, 6)$; 3 bids $(1, 0)$

No further bids are placed.

Notice that bidder 3 will always bid the price of both goods up to 1 so that bidder 1 can never win A for less than 1. In the equilibrium, bidder 1 wins A for (approximately) 1, bidder 2 wins B for (approximately) 6. Total revenue R^{sim} is (approximately) 7.

Now, observe that any strategy of bidder 1 that ever lets bidder 2 win object A is dominated, since 1 can always win A for a surplus of at least 14. If the price of A rises above 2, it is dominated for 1 to use a strategy that results in him winning only A, since he can always obtain both objects for a surplus of at least 18. Given this, it is (iteratively) dominated for bidder 2 to bid above 2 on A in the first round. Neither 1 nor 2 will ever bid more than 2 on object A, and the revenue in the AB auction is greater than the revenue in *every* equilibrium of the simultaneous auction.

[9] The FCC auction had an "activity rule" which precluded bidders from remaining dormant for too long. Incorporating such a rule would not change the results of this section. Indeed, as can be seen from the proofs, these results are robust to many rule modifications.

At this point, a few words on rule #3 of the simultaneous auction, preventing a bidder from exceeding his budget, may be in order. At first blush, it might appear that this rigs things against the simultaneous auction. After all, in the equilibrium of the sequential AB auction in Example 4, bidder 2 ends up bidding his entire budget on both goods and so, in a sense, exceeds his budget. Note, however, that this type of behavior is also possible in the simultaneous auction, and in exactly the same manner. For instance, (out of equilibrium) bidder 2 could open with a bid of 6 on good A, be outbid on this good by 1, and then bid 6 on good B. Note also that in the simultaneous auction bidder 2 never bids above 2 on good A, an amount far below his budget. Finally, the next example will assuage any lingering doubts as to whether our formulation rigs things against the simultaneous auction – for some parameter values some equilibria of the simultaneous auction are revenue superior.

Example 5 *Values:* $V^A = 9, V^B = 7, V^{AB} = 16 + \alpha$ *where* $\alpha > 0$. *Budgets:* $y_1 = 25, y_2 = 6$.

When $0 < \alpha < 3$, in the sequential AB auction bidder 2 wins A for 6, 1 wins B for 0 and $R^{AB} = 6$. In the simultaneous auction one equilibrium has bidding behavior:

Round I: 2 bids $\left(4, 1.99 - \frac{2}{3}\alpha\right)$

Round II: 1 bids $\left(4.99 + \frac{1}{3}\alpha, 2 - \frac{2}{3}\alpha\right)$

Round III: 2 bids $\left(5 + \frac{1}{3}\alpha, \cdot\right)$

No further bids are placed. Bidder 1 wins B for $2 - \frac{2}{3}\alpha$, 2 wins A for $5 + \frac{1}{3}\alpha$ and the total revenue $R^{sim} = 7 - \frac{1}{3}\alpha > R^{AB}$.

These bids are supported in the following manner. Following 2's opening bid, in *Round II* : *i*) 1 bids $\left(4.99 + \frac{1}{3}\alpha, 2 - \frac{2}{3}\alpha\right)$ if 2 opened with $\left(b_2^A, b_2^B\right) = \left(4, 1.99 - \frac{2}{3}\alpha\right)$ in *Round I*, otherwise *ii*) 1 bids $\left(5.99, b_2^B + 0.01\right)$ if $V^A - b_2^A < V^B - b_2^B$, and bids $\left(b_1^A + 0.01, 5.99\right)$ if $V^A - b_2^A \geq V^B - b_2^B$. Bidder 2's strategy in *Round III* is to just outbid bidder 1 on whichever good is higher priced. Notice that attempting to win the lower priced good would cause 2 to lose both goods. Notice also that for any opening bid of player 2, these continuation strategies yield 1 the greatest surplus he could possibly obtain, given that 2 would not let him win both goods for less than 12.

When $3 < \alpha < 5$, in the sequential AB auction bidder 1 wins both goods and revenue $R^{AB} = 12$. In the simultaneous auction, since 1 can guarantee himself a surplus of at least $4 + \alpha$ by winning both goods, it is (iteratively) dominated for 2 to open with a bid above $5 - \alpha$ on good A, as this will induce 1 to take both goods. Bidder 1 recognizes this, and neither bidder will ever bid above $5 - \alpha$ on good A. In any equilibrium of the simultaneous auction, bidder 1 wins only A for a price $p \leq 5 - \alpha$. Thus, all equilibria of the simultaneous auction have a total revenue of at most $11 - \alpha < R^{AB}$.

When $\alpha \geq 5$ bidder 1 wins both objects in either auction and the two yield the same revenue.

Thus, whereas some equilibria of the simultaneous auction are better than the equilibrium of the sequential auction when the synergies are small, this is no longer true when the synergy increases (in fact, all equilibria of the simultaneous auction are strictly inferior when $3 < \alpha < 5$).

Since $V^{AB} > V^A + V^B$ an allocation is efficient only if both objects go to the same bidder. Thus, when $3 < \alpha < 5$ the sequential *AB* is efficient whereas the simultaneous auction is not.

As noted earlier, one motivation for adopting the simultaneous auction was the feeling that it would be efficient in the presence of positive synergies. Example 5 casts some doubt on this motivation.[10]

The reasoning in Examples 4 and 5 is general and it can be shown that:

- If bidder 1 wins only object *A* or wins both *A* and *B* in the sequential *AB* auction, then the revenue from the sequential *AB* auction is at least as great as the revenue from any equilibrium of the simultaneous auction. (See Lemma 1 in the Appendix.)

The sufficient condition stated above concerns the equilibrium of the sequential *AB* auction and thus is not in terms of the primitives of the model. The next proposition restates this in terms of the primitives. This proposition follows from the above bullet point, since bidder 1 will want to win good *A* when V^A is large compared to V^B. Similarly, when the synergy α is large bidder 1 will want to win both objects.

Proposition 2 *Suppose that A and B are complements. The revenue from the sequential AB auction is no less than, and sometimes strictly greater than, that in any equilibrium of the simultaneous auction if either (a) V^A is large relative to V^B; or (b) the synergy parameter α is large enough.*

4.1 A Hybrid Form

Returning to Example 4, notice that the simultaneous auction results in each bidder winning a single object: bidder 1 wins *A* and bidder 2 wins *B*. This is the same as the equilibrium outcome of the sequential *BA* auction. In effect, the simultaneous format allows the bidders to allocate the objects among themselves in the more favorable (from their perspective) *BA* order.

Now consider a modification of the simultaneous auction with the following rules.

Rules 1 to 4 of the simultaneous format apply and in addition:

5. If there are two successive rounds with no increments in the bids, object *A* is sold to the current highest bidder and then the auction on *B* continues.

This modification accomplishes two things. First, in a sense it ultimately imposes the order *AB* upon the bidders. Second, it prevents collusive equilibria where, for instance, in the first round 1 bids one cent on good *A*, 2 bids one cent on good *B*, and they “agree” not to place any further bids, where the agreement is supported by threats to resume “normal” bidding if either player bids in the second round. We will refer to the auction using rules 1

[10] It should be emphasized, however, that there is complete information in our setting, whereas in the FCC auction the designers had to take into account incomplete information.

to 5 as the *hybrid* auction since it has elements of both the sequential and the simultaneous formats.

Let us return to Example 4 under the hybrid rules. First, for the same reasons as in the simultaneous auction, bidder 1 will win A. Once A is sold, however, now the auction on B re-opens and its price will be bid up to 6. Anticipating this, the players will bid the price of A up to 6 also. Thus, in this example, the unique equilibrium under the hybrid rules mimics the equilibrium of the AB auction.

At this stage it may appear that the hybrid auction is revenue equivalent to the AB auction. The following example (which is just Example 5 when $\alpha = 1$) shows that, in fact, it may out-perform the sequential auction.

Example 6 *Values:* $V^A = 9, V^B = 7, V^{AB} = 17$. *Budgets:* $y_1 = 25, y_2 = 6$.

In the sequential AB auction bidder 2 wins A for 6 and 1 wins B for 0. The total revenue from the sequential auction $R^{AB} = 6$.

One equilibrium of the hybrid auction has bidding behavior:

Round I: 1 bids $(5.99, 2)$; 2 bids $(4, 1.99)$.

Round II: 2 bids $(6, \cdot)$.

No further bids are placed. A is sold to bidder 2 for 6, B is sold to bidder 1 for 2. The total revenue from this equilibrium of the hybrid auction $R^{hyb} = 8 > R^{AB}$.

This equilibrium is, in fact, revenue maximal. In every equilibrium 1 wins good B, since he has more money than 2 once the B auction re-opens. Since 2 can never win B, he can be driven up to a price of 6 on A, so that 1 will never let 2 win A for less than 6. (Thus, 2 has no strategy that earns him a surplus greater than 2 and his strategy in the above equilibrium is undominated). It is (iteratively) dominated for bidder 2 to bid more than 2 on B, since bidder 1 would then win both objects for a surplus of 5, rather than win only B. Bidder 1, recognizing this, will also not bid more than 2 on good B. In every equilibrium of the hybrid game, 2 wins A for 6 and 1 wins B for at most 2.

The above equilibrium yields more than the equilibrium of the sequential auction. In fact, *every* equilibrium of the hybrid auction yields at least as much as the equilibrium of the sequential auction.[11]

The improvement the hybrid displays in Examples 4 and 6 is completely general.

Proposition 3 *Suppose that A and B are complements. The revenue from the hybrid auction is at least as great as that from the sequential AB auction.*

Efficiency The hybrid auction dominates the sequential auction in terms of revenue.[12] We now turn to the question of efficiency.

[11] Note that this result depends upon the fact that it is the auction for B that reopens, and not the auction for A. Suppose B sells first and the auction for A then reopens. In Example 4, every equilibria of this modified auction is strictly worse than the sequential auction.

[12] Proposition 2 implies that when V^A is large relative to V^B the hybrid is superior to the simultaneous auction. When V^A is small relative to V^B some equilibria of the simultaneous auction are worse than all the equilibria of the hybrid auction, but the sets may also overlap.

Recall that since we are working in a model with common values the only possible inefficiency is that each bidder is allocated one object when there are strictly positive synergies.

First, suppose that in the sequential auction the allocation is efficient, that is, both objects are allocated to bidder 1. Then the allocation in the hybrid auction is also efficient (this follows from Proposition 3). Second, suppose that in the sequential auction bidder 1 wins only A. Then it must be that bidder 1's budget is insufficient to also win B. Under these circumstances, in the hybrid auction bidder 2 will also not let bidder 1 win both objects. Finally, suppose that in the sequential auction bidder 1 wins only B. Then in the hybrid auction, as in Example 6, it is dominated for bidder 2 to raise the price of B so much that bidder 1 is better off obtaining both objects and in every equilibrium of the hybrid auction each bidder obtains one object, as in the sequential auction.

Thus the hybrid auction is equivalent to the sequential auction in terms of allocative efficiency.

5 Endogenous Budget Constraints

In this section, we consider a model where the budgets are endogenously determined rather than exogenously given. Suppose there are only two bidders and the objects are sold sequentially in some fixed order, either AB or BA, that is announced in advance. Prior to the auction, the bidders simultaneously choose budgets y_1 and y_2 which are commonly known prior to the auction and remain fixed.

As a first step, consider a situation in which only a *single* object with common value V is to be sold. Observe that there cannot be an equilibrium pair of budgets $(\bar{y}_i, \bar{y}_j)$ such that $\bar{y}_i < V$ and $\bar{y}_j < V$. Furthermore, it is dominated for any bidder to choose a budget $y_i < V$. With a single object, commitments to "small" budgets cannot arise endogenously. Now consider the following example:

Example 7 *Values:* $V^A = 10, V^B = 6, V^{AB} = 16$.

In the first stage, bidders simultaneously choose their budgets. Next, the objects are sold sequentially in the order AB. First, observe that in equilibrium both bidders cannot come with budgets exceeding $V^A + V^B = 16$. With budgets $\bar{y}_1 \geq 16$ and $\bar{y}_2 \geq 16$, neither bidder gets any surplus. If bidder 2 were instead to choose $y_2 = 9.98$, however, bidder 1 would let 2 win object A for 9.98 and then win B for free. Thus, if bidder 1 has a large budget, bidder 2 prefers to come with a "small" budget. It may be verified that it is an equilibrium for the bidders to choose (small) budgets of $\bar{y}_1 = 11$ and $\bar{y}_2 = 5$ and for bidder 2 to then win the first object for 5 and bidder 1 the second for free. The choice of a budget of 5 by 2 keeps bidder 1 indifferent between winning B and winning both objects, since either outcome results in a surplus of 6. Given 1's budget of 11, bidder 2 can do no better than the surplus of 5 he gets from a budget of 5.

Our main finding is that with *multiple* objects, in *every* equilibrium at least one bidder chooses to be budget constrained. Moreover, equilibrium payoffs are essentially unique and each good goes for less than its value. With multiple objects, commitments to "small" budgets arise endogenously. Note that bidders choose small budgets even though there are no additional transaction costs associated with choosing a larger budget. The latter may be the case, for instance, if a bidder had to obtain a line of credit from a bank and more liberal letters of credit had higher costs associated with them.

In a budget constrained equilibrium, at least one bidder cannot afford to buy both goods at their full value. We call an equilibrium *strongly budget constrained* if at least one bidder has a budget strictly less than V^A.

It is clear that if there are strong positive synergies (α is very large), the two object auction reduces to a "single" object auction in which the bundle AB is for sale. Thus, in what follows, we assume that $\alpha < V^B$. Our main result is:

Proposition 4 *The two-bidder game with endogenous budgets has a pure strategy equilibrium. Every pure strategy equilibrium is strongly budget constrained and all the equilibria are payoff equivalent (except for a relabelling of the bidders). Each good sells for less than its value.*

In order to gain some intuition as to why equilibria are strongly budget constrained, suppose that the order of sale is AB and consider the special case where $\alpha = 0$. In any equilibrium of the game with endogenous budgets, say $\left(\bar{y}_i, \bar{y}_j\right)$, each bidder must be getting a positive surplus. This is because if bidder j, say, were to choose a budget of $y_j = V^A - \varepsilon$ one of two things may happen. If $y_j > \bar{y}_i$, then certainly j will get a positive surplus. If $y_j \leq \bar{y}_i$, (at worst) bidder i will let bidder j win good A for a price of $V^A - \varepsilon$ in order to obtain good B for free (as in Example 7), since this is better for i than either winning A for $V^A - \varepsilon$, or winning both objects. Hence, in equilibrium, each bidder has a positive surplus and thus must be winning exactly one object. This means that at least one bidder has a budget smaller than $V^A + V^B$. Lemmas 5 and 6 in the Appendix provide an exact characterization of the equilibrium and show that, in fact, at least one bidder is strongly budget constrained. Proposition 4 then follows immediately.

Proposition 1 shows that in the game with endogenous budgets if the seller cannot commit to an order of sale prior to the choice of budgets, the optimal order is AB. By using Lemmas 5 and 6 in the Appendix it is easy to verify that the same is true even if the seller commits to an order of sale prior to the choice of budgets.

Flexible budgets We emphasize that Proposition 4 does not presuppose that the budgets are binding commitments on the part of the bidders. To see this, suppose that the participants in the auction are firms which send representatives with fixed budgets to bid on their behalf. Let each firm supply its agent with a cellular phone so that, for the nominal price of a phone call, the agent can call for extra funds at any time. Then the representatives' initial budgets are flexible and can be relaxed at any time. Nevertheless, Proposition

4 continues to hold; the phones have no effect on the equilibrium outcome. Consider Example 7. As discussed above, one equilibrium involves the firms dispatching their representatives to the auction with budgets of 11 and 5. The presence of cellular phones does not affect this (or any other) equilibrium. Given the equilibrium budgets and the (small) cost of obtaining additional funds, neither representative will find it profitable to call for more money after the first auction is over, or at any time during the two auctions. This reasoning is general and is a consequence of the precise equilibrium budgets derived in Lemmas 5 and 6.

Suppose there are three bidders instead of two. Then there are still strongly budget constrained equilibria, although the uniqueness result does not hold.

As an example consider the case where $\frac{1}{2}V^A < V^B$ and $V^{AB} = V^A + V^B$. Suppose the order of sale is AB. It may be verified that budget choices satisfying $\bar{y}_1 \geq V^A + V^B$, $\bar{y}_2 = V^A - \left(V^B - \frac{1}{2}V^A\right)$, and $\bar{y}_3 = \frac{1}{2}V^A$ constitute an equilibrium in undominated strategies. In the subgame following the budget selections, bidder 2 wins object A for $\bar{y}_2$ and bidder 1 wins object B for $\bar{y}_3$. However, there are also other undominated equilibria. For example, budgets satisfying $\bar{y}_1 = \bar{y}_2 = \bar{y}_3 \geq V^A + V^B$ constitute an equilibrium as well.

6 Miscellany

In this section we highlight, by means of a series of examples, some interesting phenomena that arise in multiple-object budget constrained auctions.

Revenue-enhancing constraints Suppose there are two bidders and A and B are sold in the order AB. Bidder 1 values A at 14 and B at 3. Bidder 2 values A at 8 and B at 3. Thus, we have:

Example 8 *Values:* $V_1^A = 14$, $V_2^A = 8$; $V_1^B = V_2^B = 3$. *Budgets:* $y_1 = 13$, $y_2 = 11$.

Observe that for the purposes of this example we are departing from our assumption that bidders have common values. In the absence of any budget constraints A sells for a price of 8 and B sells for 3. The total revenue to the seller is 11.

Now suppose that $y_1 = 13$ and $y_2 = 11$. Bidder 1 wants to win A at prices below 11. Bidder 2 wants to win A at prices below 8, but will force the price of A up to 11 in order to deplete 1's budget. This leaves bidder 1 with a remaining budget of 2 so that B sells for 2. Total revenue is now 13.

Thus, in this example the seller actually *benefits* from the fact that the bidders are financially constrained, a phenomenon that is impossible when there is only a single object.

Even with multiple objects, budget constraints cannot increase revenue in a common values setting, since then revenue in the unconstrained case is $V^A + V^B$, which is as large as possible given that no bidder will buy an object for more than he values it (with no synergies). When the buyers have differing valuations, however, the possibility of revenue-enhancing constraints arises. We now give some simple sufficient conditions. These are essentially that the bidders' valuations of B are similar but that their valuations of A differ, that the buyer's incomes are close to each other, and that one player is constrained with respect to his valuations.

To capture these conditions simply, suppose that $V_1^B = V_2^B \equiv V^B$, $V_1^A > V_2^A$, $y_1 = y_2 \equiv y$, and $V_1^A + V^B > y > V_2^A + V^B$. Note that were budgets unconstrained ($y > V_1^A + V^B$), the total revenue would be $V_2^A + V^B$. With the given budgets, however, bidder 1 wins A for $\min\left\{\frac{1}{2}\left(V_1^A - V^B + y\right), y\right\}$, bidder 2 wins B for $\max\left\{\frac{1}{2}\left(y - \left(V_1^A - V^B\right)\right), 0\right\}$, and the total revenue is $y > V_2^A + V^B$.

Efficiency versus revenue Our next example illustrates that in determining the order of sale there may be a trade-off between revenue and efficiency. Again, we depart from our basic model and suppose that bidders have private values.[13]

Example 9 *Values:* $V_1^A = 22$, $V_1^B = 7$; $V_2^A = 20$, $V_2^B = 8$. *Budgets:* $y_1 = 100$, $y_2 = 18$ $y_3 = 2$.

Efficiency dictates that object A go to bidder 2 and object B to bidder 1.

First, suppose the objects are auctioned in the order AB. Then bidder 2 wins A for 18 and bidder 1 wins B for 2. The total revenue to the seller is 20. The allocation is inefficient even though the environment is one of complete information.

Next, suppose the order of sale is BA. Then bidder 2 wins B for 8. Bidder 1 wins A for 10. While the total revenue is only 18, the allocation is efficient.

Thus the order AB results in a higher revenue but the allocation is inefficient. In the BA order, the situation is reversed.

Reserve prices Reserve prices are frequently employed in auctions. Two straightforward explanations for their usage are $i)$ it may be that the seller derives a positive value from the object himself and the reserve price guarantees that it will not be sold below this value and $ii)$ a reserve price can enhance revenues in circumstances where the reserve price is above second-highest valuation but the below highest.

In our setting a small reserve price may help the seller even if the seller has no value for object and even though the reserve price is below the valuations of both bidders.

Example 10 *Values:* $V^A = 10$, $V^B = 4$, $V^{AB} = 14$. *Budgets:* $y_1 = 9$, $y_2 = 6.5$

The objects are sold in the order AB. Suppose that the seller imposes a minimum selling price of $r = 1$ on B (or on both A and B). Since B then yields a surplus of at most 3, bidder 1 will win A at a price of 6.5. Bidder 2 then wins B for $y_1 - 6.5 = 2.5$. Thus, the equilibrium prices are $p^A = 6.5$ and $p^B = 2.5$, respectively. The total revenue with a reserve price of $r = 1$ is 9.

[13] In a single object auction, if the bidder who values the object the most is budget constrained, the equilibrium allocation may be inefficient. See Maskin (1992) for a discussion of this issue and some remedies.

Note that in equilibrium the reserve price is not binding. An observer presented with the results of this auction would be tempted to conclude that the imposition of a reserve price had proved irrelevant. Consider, however, the auction without the reserve price. In equilibrium, bidder 2 wins A for 6.5 and bidder 1 wins B for free. Total revenue is 6.5. Far from being irrelevant, the reserve price increases revenue by an amount greater than the reserve price itself!

Generally speaking, when there is uncertainty about the nature of the bidders, a reserve price increases revenues in some instances at the cost of sometimes preventing a beneficial trade from occurring. Example 10 shows that when there are budget constraints this problem may be mitigated by the fact that a small reserve price can have a large effect on revenue.

A non-binding reserve price can increase revenue in a variety of circumstances.

If bidder 1 does not have enough wealth to win both objects ($y_1 < 2y_2$), then when $y_1, y_2 < V^A$ and $2y_2 - y_1 < V^A - V^B < y_2$ (as in the above example), a reserve price r such that $y_2 - \left(V^A - V^B\right) < r < y_1 - y_2$ will be non-binding and enhance revenue.

If bidder 1 has enough wealth to win both objects ($y_1 > 2y_2$), then when $V^B > y_2 > \frac{V^A}{2}$, a reserve price r such that $2y_2 - V^A < r < y_2$ has the same features.

More than two objects This paper has considered two-object auctions. The analysis of auctions with more than two objects poses substantial difficulties, largely because of the large variety of phenomena that can occur in the various subgames.

We conclude with an example with three objects, A, B and C without synergies, and two conjectures regarding such auctions.

Example 11 *Values:* $V^A = 45$, $V^B = 42$, $V^C = 18$. *Budgets:* $y_1 = 21$, $y_2 = 15$.

First, suppose the objects are sold in the order ABC by means of three English auctions. In equilibrium, bidder 2 wins A for $p^A = 12$ and then bidder 1 wins both B and C, each at a price of 3. Bidder 2 gets a surplus of $V^A - p^A = 33$. If bidder 2 were to drop out of the first auction at a price of 12 then in the resulting BC auction that constitutes the subgame, the residual budgets would be $y_1 - p^A = 9$ and $y_2 = 15$, respectively. In this subgame, 2 would win B for 9 and then 1 would win C for 6. Thus by dropping out of the first auction at 12, bidder 2's surplus would be $V^B - 9 = 33$, which is the same as his surplus in equilibrium. Bidder 1, on the other hand, is better off dropping out of the first auction at $p^A = 12$ and winning B and C. The total revenue from the sequential ABC auction, $R^{ABC} = 18$.

Next, suppose that the objects are sold in the order BCA. In equilibrium, bidder 2 wins B for $p^B = 8$ and then bidder 1 wins both C and A, each at a price of 7. Bidder 2 gets a surplus of $V^B - p^B = 34$. If he were to drop out at a price of 8 then in the resulting CA auction, the residual budgets would be $y_1 - p^B = 13$ and $y_2 = 15$, respectively. In this subgame, 1 would win C for 2 and then 2 would win A for 11. Thus by dropping out of the first auction at 8, bidder 2's surplus would be $V^A - 11 = 34$, which is the same as his surplus in equilibrium. Bidder 1 is again better off dropping out of the first auction at $p^B = 8$ and winning C and A.

Thus, the total revenue from the sequential *BCA* auction, $R^{BCA} = 22$.

Example 11 shows that Proposition 1 does not generalize in a straightforward manner: with three objects it is no longer the case that the optimal order of sale is always one in which the values are declining. In the example, the order *BCA* yields more revenue than the order *ABC* (and in fact, *BCA* is the revenue maximizing order).

We make two conjectures about multiple-object auctions.

First, we conjecture that it is not optimal to sell the least valuable object C first.

Second, with three objects fix the (declining) values of B and C and consider an increase in the value of A. We conjecture that when A is valuable enough, the optimal order of sale is again one in which the values are declining. Note that for large enough V^A, all bidders want to win A. Therefore, if A is sold first it goes to bidder 1 for y_2 regardless of the order of sale of the subsequent objects. Since the price of A is unaffected by the subsequent ordering of B and C, we can deduce from Proposition 1 that the order *ABC* is superior to the order *ACB*. Thus, establishing this conjecture is equivalent to showing that if A is valuable enough it should be sold first.

Bundling and Combination Bids It is well-known that a monopoly seller may gain by selling multiple products in bundles (see, for instance, Adams and Yellen (1976)). It is easy to see that in our setting, however, bundling can only be detrimental. Selling A and B as a bundle in a single auction will fetch a price of y_2 (as long as $y_2 < V^{AB}$). Selling them separately may result in a higher revenue. In Example 2 the revenue from either order *AB* or *BA* is higher than from a single bundled auction.

A related issue concerns the possibility that bidders may bid on combinations of objects. This possibility was considered by the designers of the FCC auction but ultimately was not adopted. At least part of the reason was that, with a large number of objects, the number of possible combination bids would become computationally unmanageable (McMillan (1994)). In our model, allowing combination bids in the simultaneous auction would be detrimental in terms of revenue, largely for the same reasons that bundling would be detrimental.

7 Incomplete Information

In this paper we have considered a *complete information* model of multiple object auctions with budget constraints. In this section, we introduce an incomplete information version of the model. Our purpose is not to study the incomplete information model in detail; rather it is to indicate the issues that arise and to link the model to existing auction theory.

As before, the values of the two objects, V^A and V^B, are common and commonly known. Suppose, for the sake of simplicity, that there are only two bidders, 1 and 2, with *privately* known budgets y_1 and y_2, respectively. Specifically, suppose that the budgets are identically and independently distributed according to the distribution function F. Thus, in this model, the incomplete information pertains to the budgets and not the values.[14]

Does the order of sale *AB* still produce more revenue than the order *BA*? The basic intuition behind this result in the complete information model of Section 2 is that the wealthier bidder 1 is more prone to win the first good in the order *AB* than in the order *BA*. The seller is generally better off when 1 wins the first good and so the *AB* order is superior to the *BA* order.

This basic intuition would appear to still hold when there is incomplete information. Confirming this, however, presents substantial technical difficulties. Suppose that in the symmetric incomplete information model outlined above there is a symmetric equilibrium bidding function $\beta(y)$. Then it is very unlikely that β will be an increasing function. The reason for this is already apparent in the complete information model: in many cases it is the bidder with less wealth (bidder 2) who wins the first object (in Example 3, for instance). Thus one should not expect that the symmetric bidding strategy β in the incomplete information setting will be monotonically increasing and standard differential equation techniques used to determine equilibrium strategies do not work.[15]

Pitchik (1995) provides sufficient conditions under which there is an increasing equilibrium in which, regardless of the auction order, the wealthier bidder always wins the first object and the poorer bidder then wins the second for the residual budget of the (initially) wealthier bidder. She shows that in these circumstances, the order of sale does not affect total revenue. Under these circumstances, the order of sale is irrelevant in our complete information setting as well, confirming that at least in one simple case the results generalize. A general extension to include the cases where the order of sale matters is hampered by the need to consider non-monotonic equilibria. The accompanying technical difficulties are well-known in the literature on auctions.

8 Conclusion

The presence of budget constraints introduces several considerations into multi-object auctions. The bidders' strategic calculations are altered as they have a motivation to deplete the budgets of their rivals. On the other side of the market, the seller now desires that the wealthiest bidder win the first auction. The interaction of these two factors leads to phenomena such as the seller's preference for selling the more valuable good first and the possibility of a non-binding reserve price increasing revenue.

We have analyzed auctions in which bidders do not have enough resources to purchase *all* the items for sale at their full value. Traditionally, auction theory not only assumes that bidders have unlimited budgets but also that the price a bidder is willing to pay for one object is independent of the amount of money he has spent on other objects. In more fa-

[14] This is a special case of the model studied by Pitchik (1995) in which both values and budgets are privately known.

[15] For instance, suppose $V^A = 30$, $V^B = 20$ and the budgets are drawn from a uniform distribution on $[10, 11]$. It can be shown that in the order *AB* there is no symmetric increasing equilibrium in the first auction.

miliar terms, the traditional analysis of auctions precludes the presence of income effects. A moment's reflection will convince the reader of the implausibility of this assumption in many, if not most, settings. Consider, for instance, a sequential auction of numerous valuable paintings. The absence of income effects implies that a bidder's subsequent behavior will be identical whether he wins the first painting auctioned off for one million dollars or somehow wins it for free. More concretely, in the recent auction of National Football League television broadcasting rights, CBS won the right to broadcast AFC games for eight years for four billion dollars and effectively did not enter the subsequent competition for Monday Night Football. Undoubtedly, the fact that CBS won the AFC games lowered its valuation of Monday Night Football. Nonetheless, would CBS' bidding behavior on Monday Night Football really have been the same if it had managed to pay only one hundred million dollars for the AFC contract?

Positing budget constraints in multi-object auctions can be viewed as allowing for income effects, albeit in a very particular way. From this perspective, our analysis is a step towards a more general consideration of auctions with income effects.

A Appendix

This appendix collects various formal proofs.

A.1 Equilibrium Bidding Behavior

We first study some features of equilibrium bidding behavior when the two objects are sold sequentially by means of open ascending auctions. The basic analysis is similar to that in Pitchik and Schotter (1988) but is complicated by the presence of more than two bidders and the possibility of synergies.

For $i = 1,2$ and $j = 3-i$, let $\pi_i\left(W_i, z_i; W_j, z_j\right)$ be bidder i's equilibrium payoff in an English auction for a single object when bidder i values the object at W_i and has a budget of z_i, whereas bidder j values the object at W_j and has a budget of z_j. (Of course, the payoff $\pi_i\left(W_i, z_i; W_j, z_j\right)$ depends on bidder 3's income z_3 as well, but we fix $z_3 = y_3$ and economize on notation by suppressing the dependence of π_i on z_3.) The functions π_i determine equilibrium payoffs in the second auction. W_i differs from W_j when the synergy is non-zero.

Note that *i*) $\pi_i\left(W_i, z_i; W_j, z_j\right)$ is non-decreasing in W_i and z_i; *ii*) $\pi_i\left(W_i, z_i; W_j, z_j\right)$ is non-increasing in W_j and z_j; and *iii*) if $\min\{W_i, z_i\} < \min\left\{W_j, z_j\right\}$ then bidder i loses the auction and so $\pi_i\left(W_i, z_i; W_j, z_j\right) = 0$.

In what follows, we denote by I the object that is sold in the first auction and by II the object that is sold in the second auction. Thus, for example, if the objects are sold in the order AB then $I = A$ and $II = B$.

Define:

$$\widehat{p}_i^I = \sup\left\{p : \left(V^I - p\right) + \pi_i\left(V^{II} + \alpha, y_i - p; V^{II}, y_j\right) > \pi_i\left(V^{II}, y_i, V^{II} + \alpha, y_j - p\right)\right\}.$$

For any $p < \widehat{p}_i^I$, the left hand side of the inequality is bidder i's payoff if he wins the first auction at p and the right hand side is bidder i's payoff if he drops out of the first auction at p and as a result, bidder j wins the first auction at p.[16] Of course, if $p < y_3$ then even if bidder i were to drop out of the first auction at p, bidder 3 would bid the price that bidder j pays up to y_3. Define:

$$\overline{p}_i^I = \max\left\{\widehat{p}_i^I, y_3\right\}.$$

The following properties of $\overline{p}_i^I$ will be useful below.

Property 1 $\overline{p}_i^A \geq \overline{p}_i^B$.

This is quite intuitive and just says that each bidder is willing to bid more for the first object when it is A than when it is B.

Property 2 Either $\overline{p}_1^I \leq \overline{p}_2^I \leq V^I$ or $\overline{p}_1^I \geq \overline{p}_2^I \geq V^I$.

The second property delineates two cases. If each bidder wins one object then bidder 2 is willing to bid higher for the first object than bidder 1. This is because bidder 1's residual budget in the second auction after winning the first is greater than bidder 2's in similar circumstances. Thus, the surplus from winning just the second object is less for bidder 2 than for bidder 1, and so 2 is willing to bid higher on the first. In the second case, since the players want to bid above V^I they must be planning to win both goods (hence $\alpha > 0$). Since, after winning the first good, 1 would have more money left over to bid on the second good than would 2 in the same position, 1 is willing to bid higher. Note that in Example 4, bidder 1 wins good A (and B). When 1 wins the first good for p^I in a sequential auction, it is because 2 runs up against his budget constraint. Thus, 1 can never be forced to pay more than p^I for good I and it is a short step to see that the sequential auction is revenue maximizing.

(Formal proofs of these properties are omitted but available from the authors.)

Define

$$p_i^I = \min\left\{\overline{p}_1^I, y_i\right\}. \tag{1}$$

Property 1 then implies that:

$$p_i^A \geq p_i^B. \tag{2}$$

If $p_i^I > y_3$, bidder i will never drop out of the bidding for object I at prices below p_i^I. Bidder 3 will always bid the price of I up to y_3. If bidder 1 stops bidding below y_3, bidder 2 will stay in to win I at y_3, since otherwise he will not win either object. We have,

1. If $p_1^I < p_2^I$, bidder 2 wins object I. Bidder 1 (or bidder 3) forces the price in the first auction up to $\min\left\{p_2^I, \max\left\{p_1^I, y_2 - y_3\right\}\right\}$ and bidder 2 pays that price. Note that at prices beyond p_1^I, bidder 1 does not want to win object I, and is bidding only to deplete 2's income. At prices above $y_2 - y_3$ there is no incentive to further deplete 2's income since the price of object II will not fall below y_3 in any case.

[16] The supremum is well defined since $p = 0$ satisfies the inequality and so the set is not empty. Furthermore, $p = V^I + \max\{\alpha, 0\}$ is an upper bound of the set.

2. If $p_1^I > p_2^I$ bidder 1 wins object I. From Property 2, if $V^I \geq p_1^I$ then $p_2^I = y_2$. In that case, bidder 2 runs up against his budget constraint even though he would like to continue bidding higher, and thus *bidder* 1 *wins object I* for $p_2^I = y_2$. If $V^I < p_1^I$ then even if 1 wins object I at a price below p_1^I, he still has enough money left over to win object II. Therefore, 2 has no incentive to try and push the price beyond p_2^I and 1 wins object I for p_2^I.

3. If $p_1^I = p_2^I$ then *either bidder* 1 *or bidder* 2 *wins object I* for $p_1^I = p_2^I$.

The equilibrium price of object I can thus be written as:

$$p^I = \min\left\{p_2^I, \max\left\{p_1^I, y_2 - y_3\right\}\right\}. \tag{3}$$

Since $p_i^A \geq p_i^B$, we have $p^A \geq p^B$, that is, the equilibrium price of the first object in the order AB is at least as high as in the order BA.

A.2 Proof of Proposition 1

We break up the proof into three separate claims. To shorten the proof, we restrict our attention to the case $y_1 > y_2 > y_3$, $V^B > y_3$. We also assume that $y_2 < V^A + V^B$, since the reverse inequality is the trivial unconstrained case.

Define $\alpha_+ = \max\{\alpha, 0\}$ and $\alpha_- = \min\{\alpha, 0\}$. Object X is worth at most $V^X + \alpha_+$ and at least $V^X + \alpha_-$.

Proof. We break up the proof into three separate claims.

CLAIM 1: *If bidder* 1 *wins only object B in the order BA, then the revenue from order AB is at least as large as that from BA.*

Notice that since each bidder wins one object each, $p^B \leq V^B$.

Since bidder 1 wins B in the order BA it must be that $p_1^B \geq p_2^B$.

If $p_1^B > p_2^B$, bidder 1 wins object B for y_2. However, bidder 1 would rather drop out at y_2 and win object A for y_3 in the second auction than win only object B for y_2. Therefore, it must be that $p_1^B = p_2^B = p^B < y_2$.

Suppose $p^B = V^B$. Since in this case bidder 1 would just as soon win B for a surplus of 0 as drop out at $p = V^B$, it must be that $y_2 - V^B \geq V^A$ and $\alpha \geq 0$, so that $y_2 \geq V^A + V^B$ and the total revenue is $V^A + V^B$ in the order BA and at least this much in the order AB.

Suppose $p_1^B = p_2^B = p^B < V^B$. If $\widehat{p}_1^B < y_3$, then bidder 2 wins B for y_3, so we must have $\widehat{p}_1^B \geq y_3$ and so $p_i^B = \widehat{p}_1^B$. From the definition of p_i^B, for $i = 1,2$, $V^B - p^B = V^A - \max\left\{\min\left\{V^A + \alpha_-, y_i - p^B\right\}, y_3\right\}$. Since $y_1 > y_2$, $\max\left\{\min\left\{V^A + \alpha_-, y_i - p^B\right\}, y_3\right\} = \min\left\{V^A + \alpha_-, y_3\right\}$. But since $p^B \geq y_3$ and $V^A > V^B$ we cannot have $V^B - p^B = V^A - y_3$, so this case is ruled out. Thus, $p^B = V^B + \alpha_-$ and the price of the second object is $V^A + \alpha_-$. Since $V^A + \alpha_- \leq y_i - p^B$,we have $y_i \geq \left(V^A + \alpha_-\right) + \left(V^B + \alpha_-\right)$, and in the order AB revenue is at least $\left(V^A + \alpha_-\right) + \left(V^B + \alpha_-\right)$. □

CLAIM 2: *If bidder* 1 *wins only object A in the order BA*, *then the revenue from the order AB is at least as large as that from the order BA.*

If one bidder wins both objects in the order *AB*, then total revenue is

$$\min\left\{\max\left\{V^A, \min\left\{V^A+\alpha_+, y_2-V^B\right\}\right\}, y_2\right\}+\min\left\{V^B, y_2\right\},$$

whereas if bidder 1 wins A alone, the total revenue is at most $\min\left\{V^A, y_2\right\}+\min\left\{V^B, y_2\right\}$. Therefore, suppose that in both orders each bidder wins one object.

In the order *BA*, bidder 1 pays $\max\left\{\min\left\{V^B+\alpha, y_2-p^B\right\}, y_3\right\}$ for A.

If bidder 1 pays $V^A+\alpha_-$ for A in the order *BA*, then $y_2-\left(V^B+\alpha_-\right) \geq \left(V^A+\alpha_-\right)$. Since $y_1 \geq y_2$, in both orders total revenue is $\left(V^A+\alpha_-\right)+\left(V^B+\alpha_-\right)$. In the order *AB*, if B sells for $V^B+\alpha_-$ then bidder 2 must have bid the price on A up to $\min\left\{V^A+\alpha_-, y_2\right\}$ and revenue is $\min\left\{V^A+\alpha_-, y_2\right\}+\left(V^B+\alpha_-\right)$. In the order *BA*, object A can be bid up to at most $\min\left\{V^A+\alpha_-, y_2\right\}$, so that revenue is at most $\left(V^B+\alpha_-\right)+\min\left\{V^A+\alpha_-, y_2\right\}$.

If bidder 1 pays $\max\left\{y_2-p^B, y_3\right\}$ for A in the order *BA*, then the total revenue is $p^B+\max\left\{y_2-p^B, y_3\right\}$. If bidder i wins only A in the order *AB* total revenue is $p^A+\max\left\{y_i-p^A, y_3\right\}$. Since $p^B \leq p^A$ and $y_2 \leq y_i$, revenue is at least as large in the order *AB* as in the order *BA*. □

CLAIM 3: *If bidder* 1 *wins both objects when the order is BA*, *then the revenue from the order AB is the same as that from the order BA.*

The total revenue when bidder 1 wins both objects in the order *BA* is:

$$R^{BA}=\min\left\{\max\left\{V^B, \min\left\{V^B+\alpha_+, y_2-V^A\right\}\right\}, y_2\right\}+\min\left\{V^A, y_2\right\} \tag{4}$$

and the total revenue when bidder 1 wins both objects in the order *AB* is:

$$R^{AB}=\min\left\{\max\left\{V^A, \min\left\{V^A+\alpha_+, y_2-V^B\right\}\right\}, y_2\right\}+\min\left\{V^B, y_2\right\}. \tag{5}$$

It is easy to verify that if $V^A+V^B+\alpha_+ \geq y_2$ then either $R^{BA}=R^{AB}=y_2$ or $R^{BA}=R^{AB}=\min\left\{V^A, y_2\right\}+\min\left\{V^B, y_2\right\}$. If $V^A+V^B+\alpha_+<y_2$ then $R^{BA}=R^{AB}=V^A+V^B+\alpha_+$.

The gain to bidder 1 from dropping out of the first auction in the order *BA* at the price p^B is, by definition, $\pi_1\left(V^A, y_1; V^A+\alpha, y_2-p^B\right)$. Similarly, the gain to bidder 1 from dropping out of the first auction in the order *AB* at the price p^A is $\pi_1\left(V^A, y_1; V^A+\alpha, y_2-p^B\right)$.

It can be argued that[17]

$$\pi_1\left(V^A, y_1; V^A+\alpha, y_2-p^B\right) \geq \pi_1\left(V^B, y_1; V^B+\alpha, y_2-p^A\right). \tag{6}$$

The fact that bidder 1 wins both objects in the order *BA* implies that

$$V^A+V^B+\alpha-R^{BA} \geq \pi_1\left(V^A, y_1; V^A+\alpha, y_2-p^B\right)$$

[17] A formal proof of this step is available from the authors.

and so

$$V^A + V^B + \alpha - R^{AB} \geq \pi_1\left(V^B, y_1; V^B + \alpha, y_2 - p^A\right)$$

so that bidder 1 would rather win both objects in the order *AB* also.

Hence the revenue in both orders is the same. □

Claims 1 to 3 complete the proof. ■

A.3 Proofs of Propositions 2 and 3

We begin by establishing two results that show revenue ranking of the sequential auction versus the simultaneous auction depends on the equilibrium outcome of the sequential auction.

Lemma 1 *Suppose* $\alpha \geq 0$. *If bidder* 1 *wins only object A or wins both A and B in the sequential AB auction, then the revenue from the sequential AB auction is at least as great as the revenue from any equilibrium of the simultaneous auction.*

Proof. First, suppose bidder 1 wins both objects in the sequential *AB* auction. Then the total revenue is:

$$R^{AB} = \min\left\{\max\left\{V^A, \min\left\{V^A + \alpha, y_2 - V^B\right\}\right\}, y_2\right\} + \min\left\{V^B, y_2\right\}$$

so that:

$$R^{AB} = \begin{cases} \min\left\{V^A, y_2\right\} + \min\left\{V^B, y_2\right\} & \text{if } y_2 \leq V^A + V^B \\ y_2 & \text{if } V^A + V^B < y_2 \leq V^A + V^B + \alpha \\ V^A + V^B + \alpha. & \text{if } V^A + V^B + \alpha < y_2. \end{cases}$$

If $y_2 \leq V^A + V^B$, then the revenue in the simultaneous auction cannot be greater than R^{AB} . This is because it is dominated for bidder 1 to let bidder 2 wins both objects and thus, in neither auction will bidder 2 bid above the value. Similarly, if $V^A + V^B + \alpha < y_2$, then R^{AB} is maximal.

We show that R^{AB} is also maximal when $V^A + V^B < y_2 \leq V^A + V^B + \alpha$. Suppose not. Then there is an equilibrium of the simultaneous auction in which the total revenue exceeds $y_2 > V^A + V^B$. Thus one of the bidders, say 1, must be winning both objects. The most that bidder 2 could bid on A is $y_2 - V^B$ and the most he could bid on B is $y_2 - V^A$. First, suppose that in the first round, bidder 2 is bidding less than V^X on some object X. Then bidder 1 could bid $y_2 - V^X$ on the other object, win it at that price and then win X for V^X. Next, suppose that in the first round bidder 2 is bidding $q^A > V^A$ on object A and $q^B > V^B$ on object B. But now bidder 1 can bid (slightly above) $y_2 - q^B$ on object A and (slightly above) q^B on object B. This will cause bidder 2 to drop out of both auctions and thus bidder 1 will win both auctions for a total of y_2. In either case the total revenue in the simultaneous auction does not exceed y_2.

Thus, we have shown that if bidder 1 wins both objects in the sequential *AB* auction the revenue from the simultaneous auction cannot be greater than R^{AB}.

Next, suppose that bidder 1 wins only *A* in the sequential *AB* auction. We now show that each bidder's surplus in the simultaneous auction must be at least as large as that in the sequential auction.

Suppose bidder 1 opens the simultaneous auction with a bid of p^A for *A* and 0 for *B*. If $p^A = y_2$ then 1 wins good *A* thereby obtaining a surplus at least as large as in the sequential auction. If $p^A < y_2$ then $V^A - p^A = V^B - \min\{V^B, y_3\}$ and $y_3 > y_1 - p^A$, or else 2 would have bid beyond p^A in the sequential auction. Therefore, if 2 pushes the bid beyond p^A in the simultaneous auction, 1 can drop out of the bidding for *A* and win good *B* for $\min\{V^B, y_3\}$. In both cases, bidder 1 guarantees himself a surplus at least as large as his surplus in the sequential auction.

Suppose bidder 2 opens the simultaneous auction with a bid of p^A for *A* and 0 for *B*. If bidder 1 lets 2 win *A* for this price, then 2's surplus, $V^A - p^A$ is at least as large as in the sequential auction. If bidder 2 does not win *A* then he can win *B* at a price not exceeding that in the sequential auction since by outbidding bidder 2 on *A* bidder 1's residual budget is smaller.

Since each bidder's surplus in the simultaneous auction is at least as large as in the sequential auction, the revenue cannot be larger.

Thus we have shown that when bidder 1 wins only object *A* in the sequential *AB* auction the revenue from the simultaneous auction is no greater than R^{AB}.

This completes the proof. ■

Proof of Proposition 2

We first show that the sequential auction is revenue superior if (a) holds.

Fix V^B. For large enough V^A, $V^A - y_2 > V^B - y_3$ and bidder 1 prefers winning *A* in the order *AB* to winning only *B*. From Lemma 1, $R^{AB} \geq R^{sim}$.

We now show that the sequential auction is revenue superior if (b) holds and that as α increases from 0 the inequality may strict for equilibria of the simultaneous auction.

Suppose $y_1 > 2y_2$, and that when $\alpha = 0$, 1 wins only one good in the *AB* auction. Let $\overline{\alpha}$ satisfy

$$V^A + V^B + \overline{\alpha} - 2y_2 = V^B - y_3.$$

For all $\alpha > \overline{\alpha}$ bidder 1 wins both goods in the *AB* auction and $R^{AB} = y_2 + \min\{V^B, y_2\}$. Let $\widehat{\alpha}$ solve:

$$V^A + V^B + \widehat{\alpha} - 2y_2 = V^A - y_3.$$

Suppose that $\widehat{\alpha} > \alpha > \overline{\alpha}$. In one equilibrium of the simultaneous auction, in the first round 2 bids y_2 on good *B* and 1 bids (just below) y_2 on good *B* and (just above) y_3 on good *A* . 2 cannot profitably deviate and win good *A*, since $\alpha > \overline{\alpha}$ so that 1 prefers winning both goods to winning *B* for y_2. Bidder 1 cannot profitably deviate by winning both goods since $\widehat{\alpha} > \alpha$. Let $\overline{p}$ solve

$$V^A + V^B + \overline{\alpha} - 2y_2 = V^A - \overline{p}.$$

Note that $\overline{p} < y_2$. If the price of good A ever rises above $\overline{p}$ in the simultaneous auction, then 1 will win both goods since this is better than winning either A for $\overline{p}$ or B for y_3. Therefore, it is dominated for 2 to bid above $\overline{p}$ on A and every equilibrium of the simultaneous auction earns no more than $\overline{p} + \min\left\{V^B, y_2\right\} < y_2 + \min\left\{V^B, y_2\right\} = R^{AB}$.

If bidder 1 wins both goods in the AB auction when $\alpha = 0$, he will still win both goods when $\alpha > 0$, and from Lemma 1 the AB auction is always at least as good as the simultaneous auction. ■

Proof of Proposition 3

The proposition follows from Lemmas 2 and 3 below.

Lemma 2 *Suppose* $\alpha \geq 0$. *If bidder* 1 *wins only object A or wins both A and B in the sequential AB auction, then the revenue from the sequential AB auction is the same as that in the hybrid auction.*

Proof. Suppose player 1 wins only A for p^A. Then bidder 1's surplus in the AB game is $V^A - p^A \geq V^B - \min\left\{V^B, \max\left\{y_2 - p^A, y_3\right\}\right\}$ and bidder 2's surplus is $V^B - \min\left\{V^B, \max\left\{y_1 - p^A, y_3\right\}\right\}$, where $V^A - p^A \geq V^B - \min\left\{V^B, \max\left\{y_1 - p^A, y_3\right\}\right\}$. Given that the hybrid auction is followed by an auction for B once the bidding stops, neither player will follow a strategy which involves dropping out of the bidding for A at a price $p < p^A$, since this yields at most $V^B - \min\left\{V^B, \max\{y_2 - p, y_3\}\right\} \leq V^A - p^A < V^A - p$. On the other hand, neither player will bid more that p^A for A either, or else he would have done so in the sequential auction. Thus, A will also sell for p^A in the hybrid auction and the revenue will be the same as in the hybrid auction.

Similarly, if 1 wins A for P^A in the AB game, and then goes on to win B as well, A will again sell for P^A in the hybrid game and revenues will be the same. ■

Lemma 3 *Suppose* $\alpha \geq 0$. *If bidder* 1 *wins only object B in the sequential AB auction, then the revenue from the sequential AB auction is no greater than the revenue from any equilibrium of the hybrid auction.*

Proof. First, consider the sequential AB auction.

Since bidder 1 wins only B in the sequential AB auction, $p_1^A \leq p_2^A$ and bidder 2 wins A for $\min\left\{p_2^A, \max\left\{y_2 - y_3, p_1^A\right\}\right\}$.

If $y_2 > V^A + V^B$ and $\alpha > 0$ then both objects would have been won by the same bidder and thus either $y_2 \leq V^A + V^B$ or $\alpha = 0$.

1. If $\alpha = 0$ and $y_2 > V^A + V^B$ then $p^A = V^A$, $p^B = V^B$ and $R^{AB} = V^A + V^B$, which is maximal in this case. If $y_2 = V^A + V^B$ then, for $\alpha \geq 0$, again $R^{AB} = V^A + V^B$.

2. If $y_2 < V^A + V^B$, then $p^B < V^B$ and the total revenue

$$\begin{aligned} R^{AB} &= \min\left\{p_2^A, \max\left\{y_2 - y_3, p_1^A\right\}\right\} \\ &+ \max\left\{y_2 - \min\left\{p_2^A, \max\left\{y_2 - y_3, p_1^A\right\}\right\}, y_3\right\}. \end{aligned}$$

(a) If $p_1^A \geq y_2 - y_3$, then $\min\left\{p_2^A, \max\left\{y_2 - y_3, p_1^A\right\}\right\} = p_1^A$ and since $V^B > y_3 \geq y_2 - p_1^A$ we have $R^{AB} = p_1^A + y_3$.

(b) If $p_1^A < y_2 - y_3$ then $\min\left\{p_2^A, \max\left\{y_2 - y_3, p_1^A\right\}\right\} = \min\left\{p_2^A, y_2 - y_3\right\}$. So the total revenue is $R^{AB} \leq y_2$.

Now consider the hybrid auction:

1. If $\alpha = 0$ and $y_2 \geq V^A + V^B$ then the object that is sold second goes for its value and therefore so does the first and $R^{sim} = V^A + V^B$. If $\alpha > 0$ and $y_2 = V^A + V^B$ then again total revenue in the simultaneous auction is $R^{sim} = V^A + V^B$. In both of these cases, $R^{sim} = R^{AB}$.

2. If $y_2 < V^A + V^B$ and bidder 1 wins both objects, then revenues are $\min\left\{V^A, y_2\right\} + \min\left\{V^B, y_2\right\}$ which are maximal. So suppose each bidder wins one object.

   (a) If $p_1^A \geq y_2 - y_3$, then by definition of p_i^A, both bidders prefer winning object A for p_1^A, to winning object B for $y_3 \geq y_2 - p_1^A$. Since bidder 3 will not let B sell for less than y_3, $R^{sim} \geq p_1^A + y_3$.

   (b) If $p_1^A < y_2 - y_3$, let I be the first object to sell in the simultaneous auction, q^I its selling price, and let i denote the bidder that wins I. The second object, II, sells for $q^{II} = \min\left\{V^{II}, \max\left\{y_j - q^I, y_3\right\}\right\}$. If $q^{II} = V^{II}$, then $q^I = y_2$ and $R^{sim} \geq y_2$. If $q^{II} = \max\left\{y_j - q^I, y_3\right\}$, then $R^{sim} = q^I + \max\left\{y_j - q^I, y_3\right\} \geq y_2$.

This completes the proof. ■

A.4 Proof of Proposition 4

We first establish some properties of any equilibrium of the game with endogenous budgets.

Lemma 4 *In any equilibrium with budgets $(\bar{y}_i, \bar{y}_j)$, where $\bar{y}_j \leq \bar{y}_i$:*
(a) each bidder wins exactly one object X, for a price less than $V^X + \alpha_-$;
(b) bidder j wins the second object; and
(c) the first object sells for $\bar{y}_j$.

Proof. (a) We show that for small enough ε, a choice of $y_j = V^I + \alpha_- - \varepsilon$, earns j a positive surplus in the auction subgame, for any value of $\bar{y}_i$. If $V^I + \alpha_- - \varepsilon > \bar{y}_i$, j can win object I. If $V^I + \alpha_- - \varepsilon \leq \bar{y}_i$, at worst i will let j win I for $V^I + \alpha_- - \varepsilon$, since this will yield i a surplus of V^{II}, while winning only I would net $\varepsilon - \alpha_- < V^{II}$ for small enough ε, and winning both objects would net i

$$
\begin{aligned}
& V^I + V^{II} + \alpha - \left(V^I + \alpha_- - \varepsilon\right) - \min\left\{V^{II}, V^I + \alpha_- - \varepsilon\right\} \\
= \;& \left(V^{II} + \varepsilon - \alpha_-\right) + \left(\alpha - \min\left\{V^{II}, V^I + \alpha_- - \varepsilon\right\}\right) \\
< \;& V^{II}
\end{aligned}
$$

for small enough ε.

(b) From (a) we know that i is winning exactly one object for less than $V^X + \alpha_-$. Suppose $\bar{y}_j < \bar{y}_i$ and i is winning the first object. Since $\bar{y}_j < \bar{y}_i$ and $y_3 = 0$, if i likes winning object I at p^I at least as much as dropping out and winning the second object, j must strictly prefer winning I. Since j concedes I to i, it must be that he is at his budget constraint so that $p^I = \bar{y}_j$. Now, j would be better off choosing a slightly bigger budget to make i either spend more on the object, or concede I to him, contradicting the fact that $(\bar{y}_i, \bar{y}_j)$ constitute equilibrium budget choices. Therefore, i must be winning the second object (and without loss of generality, can be assumed to be winning the second object when $\bar{y}_j = \bar{y}_i$). This establishes the claim.

(c) From (b) we know that j wins I. Let p^I denote the price paid by bidder j for I and suppose that $p^I < \bar{y}_j$. On the one hand, j is willing to bid up to p^I. On the other hand, since from (a) the price of the second object is less than $V^{II} + \alpha_-$, if bidder i is not pushing j any higher than p^I it must be that he cannot. Thus, j must be just indifferent between winning I at a price p^I and winning II at a price $(\bar{y}_i - p^I)$. Therefore,

$$V^I - p^I = V^{II} - \left(\bar{y}_i - p^I\right). \tag{7}$$

But if instead, j chose a budget $y_j > V^I + V^{II}$, i would bid up to $\min\{V^I, \bar{y}_i\}$ on the first object and j would earn $V^{II} - (\bar{y}_i - \min\{V^I, \bar{y}_i\})$ on the second, which is more than (7). Therefore, $p^I = \bar{y}_j$. ■

In proving Proposition 4, we consider the case of substitutes (Lemma 5 below) and complements (Lemma 6) separately.

A.4.1 Substitutes

Lemma 5 *Suppose* $\alpha \leq 0$. *The budgets* $(\bar{y}_i, \bar{y}_j)$ *and the prices* $(\bar{p}^I, \bar{p}^{II})$ *constitute an equilibrium outcome of the game with endogenous budgets if:*

$$\begin{array}{lll} \bar{y}_i \geq V^I - V^{II} & \bar{p}^I = V^I - V^{II} & \textit{if } \frac{1}{2}V^I \geq V^{II} + \frac{1}{2}\alpha \\ \bar{y}_j = V^I - V^{II} & \bar{p}^{II} = 0 & \\ & & \\ \bar{y}_i \geq \frac{1}{2}V^I + V^{II} + \frac{3}{2}\alpha & \bar{p}^I = \frac{1}{2}\left(V^I + \alpha\right) & \textit{if } \frac{1}{2}V^I < V^{II} + \frac{1}{2}\alpha \\ \bar{y}_j = \frac{1}{2}\left(V^I + \alpha\right) & \bar{p}^{II} = 0 & \end{array}$$

All equilibria result in the same payoffs (except for a relabelling of the bidders).

Proof. Considering the two cases separately, we first verify that $(\bar{y}_i, \bar{y}_j)$ constitute equilibrium budget choices.

Case 1: $\frac{1}{2}V^I \geq V^{II} + \frac{1}{2}\alpha$. Given budgets $\bar{y}_i \geq V^I - V^{II}$ and $\bar{y}_j = V^I - V^{II}$, it is an equilibrium outcome of the auction subgame for bidder j to win I for a price of $V^I - V^{II}$ and for bidder i to win II for free. The equilibrium payoff of bidder j is V^{II} and that of i is also V^{II}.

Now, suppose $\bar{y}_i \geq V^I - V^{II}$. If bidder $j \neq i$ wins both objects then his payoff is less than

$$\begin{aligned} & (V^I + V^{II} + \alpha) - (V^I - V^{II}) - \min\left\{V^{II}, V^I - V^{II}\right\} \\ = \;& 2V^{II} + \alpha - \min\left\{V^{II}, V^I - V^{II}\right\} \end{aligned}$$

and for either realization of the minimum, this is no greater than V^{II}. If j wins only the second object his payoff is at most V^{II}. Finally, i will not let j win the first object alone for less than $V^I - V^{II}$. Thus, bidder j cannot profitably deviate if $\bar{y}_i \geq V^I - V^{II}$. Similarly, if $\bar{y}_j = V^I - V^{II}$, bidder i cannot profitably deviate.

Case 2: $\frac{1}{2}V^I < V^{II} + \frac{1}{2}\alpha$. With budgets $\bar{y}_i \geq \frac{1}{2}V^I + V^{II} + \frac{3}{2}\alpha$ and $\bar{y}_j = \frac{1}{2}\left(V^I + \alpha\right)$, it is an equilibrium outcome of the auction subgame for bidder j to win I for a price of $\frac{1}{2}\left(V^I + \alpha\right)$ and for bidder i to win II for free. The equilibrium payoff of bidder i is V^{II} and that of bidder j is $\frac{1}{2}V^I - \frac{1}{2}\alpha$.

Suppose $\bar{y}_i \geq \frac{1}{2}V^I + V^{II} + \frac{3}{2}\alpha$. If bidder j chooses a $y_j < \frac{1}{2}\left(V^I + \alpha\right)$, then i will win both objects. Therefore, suppose that $y_j > \frac{1}{2}\left(V^I + \alpha\right)$. If j wins I and only I then he pays at least $\frac{1}{2}\left(V^I + \alpha\right)$. (This is because bidder i can force the price up to $\frac{1}{2}\left(V^I + \alpha\right)$ when j's budget is $\frac{1}{2}\left(V^I + \alpha\right)$, and i can still force the price to at least this level when j's budget is larger.) If bidder j wins only II then he pays at least

$$\begin{aligned} \min\{V^{II} + \alpha, \bar{y}_i - \left(V^I + \alpha\right)\} \;&\geq\; \min\{V^{II} + \alpha, V^{II} + \frac{1}{2}\alpha - \frac{1}{2}V^I\} \\ &=\; V^{II} + \frac{1}{2}\alpha - \frac{1}{2}V^I \end{aligned}$$

since $\alpha > -V^I$ and so his net gain is no greater than $\frac{1}{2}V^I - \frac{1}{2}\alpha$. Bidder j can win both objects only if $\alpha = 0$ and this then nets him 0 since $\bar{y}_i \geq \max\{V^I, V^{II}\}$. Thus $\bar{y}_j = \frac{1}{2}\left(V^I + \alpha\right)$ is a best response to any $\bar{y}_i \geq \frac{1}{2}V^I + V^{II} + \frac{3}{2}\alpha$.

Suppose $\bar{y}_j = \frac{1}{2}\left(V^I + \alpha\right)$ and bidder i chooses a $y_i \neq \bar{y}_i$. If i wins both objects it must be that $\alpha = 0$ and then he pays a total of $2\bar{y}_j = V^I$ for a net gain of V^{II}, which is his equilibrium payoff. If he wins only I, then this is a profitable deviation only if I is obtained at a price below $V^I - V^{II} < \frac{1}{2}V^I + \frac{1}{2}\alpha$. But i cannot obtain I at such a price, since bidder j would continue to raise the bid. Finally, i is already winning II for a price of 0. Thus $\bar{y}_i \geq \frac{1}{2}V^I + V^{II} + \frac{3}{2}\alpha$ is a best response to $\bar{y}_j$.

Thus, we have shown that in both cases $(\bar{y}_i, \bar{y}_j)$ as specified are equilibrium budget choices. The fact that in each case the equilibrium payoffs are the same from all such equilibria is immediate.

Suppose $(\bar{y}_i, \bar{y}_j)$ are equilibrium budget choices with $\bar{y}_j \leq \bar{y}_i$. Now, since j is not choosing a budget $y_j < \bar{y}_j$, it must be the case that doing so causes him to not win object I, since if he still won I it would perforce be at a price below $\bar{y}_j$, and from Lemma 4 (c), this is better for him. Thus, in the subgame with budgets $(\bar{y}_i, y_j)$ either :

(i) there is an equilibrium such that bidder i wins the first object instead of the second; or

(ii) there is an equilibrium such that bidder i wins both objects.

CLAIM 1: (i) *holds only if* $\frac{1}{2}V^I \geq V^{II} + \frac{1}{2}\alpha$ *and* $\bar{y}_j = V^I - V^{II}$.

If any choice of $y_j < \bar{y}_j$ causes i to win the first object instead of the second, it must be that when j chooses a budget of $\bar{y}_j$, bidder i is just indifferent between winning II and winning I. Hence, $V^{II} = V^I - \bar{y}_j$ or equivalently, $\bar{y}_j = V^I - V^{II}$ and by construction $\bar{y}_i \geq \bar{y}_j = V^I - V^{II}$.

Since bidder i does not prefer to win both objects rather than just II, we have:

$$\begin{aligned} V^I + V^{II} + \alpha - \bar{y}_j - \min\left\{V^{II}, \bar{y}_j\right\} &\leq V^{II} \\ \alpha + V^{II} - \min\left\{V^{II}, V^I - V^{II}\right\} &\leq 0 \end{aligned}$$

If $V^{II} \leq V^I - V^{II}$ then $\alpha \leq 0$ implies that $\frac{1}{2}V^I \geq V^{II} + \frac{1}{2}\alpha$. If $V^{II} > V^I - V^{II}$ then again $\frac{1}{2}V^I \geq V^{II} + \frac{1}{2}\alpha$. Thus in either case we have the necessary condition that $\frac{1}{2}V^I \geq V^{II} + \frac{1}{2}\alpha$. This establishes the claim. □

CLAIM 2: (ii) *holds only if* $\frac{1}{2}V^I \leq V^{II} + \frac{1}{2}\alpha$ *and* $\bar{y}_j = \frac{1}{2}\left(V^I + \alpha\right)$.

If any choice of $y_j < \bar{y}_j$ causes i to win both objects instead of the second, it must be that when j chooses a budget of $\bar{y}_j$, bidder i is just indifferent between winning II and winning both objects. Hence, $V^{II} = V^I + V^{II} + \alpha - 2\bar{y}_j$, or equivalently,

$$V^{II} = V^I + V^{II} + \alpha - \bar{y}_j - \min\left\{V^{II}, \bar{y}_j\right\}$$

Now there are two cases to consider.

If $\bar{y}_j > V^{II}$, then $\bar{y}_j = V^I - V^{II} + \alpha$ and thus

$$\frac{1}{2}V^I > V^{II} - \frac{1}{2}\alpha.$$

Since by choosing a y_i large enough, bidder i could have won object I at a price of $\bar{y}_j = V^I - V^{II} + \alpha$, it must be that this is no better than winning object II for free. Thus $V^{II} \geq V^I - \left(V^I - V^{II} + \alpha\right)$ or equivalently, that $\alpha \geq 0$. Since we have assumed $\alpha \leq 0$ this can only happen if $\alpha = 0$. But then this is the same as the case considered in Claim 1.

If $\bar{y}_j \leq V^{II}$, then $\bar{y}_j = \frac{1}{2}\left(V^I + \alpha\right)$.

Since by choosing a y_i large enough, bidder i could have won object I at a price of $\bar{y}_j = \frac{1}{2}\left(V^I + \alpha\right)$, it must be that this is no better than winning object II for free. Thus $V^{II} \geq \frac{1}{2}V^I - \frac{1}{2}\alpha$ or equivalently,

$$\frac{1}{2}V^I \leq V^{II} + \frac{1}{2}\alpha.$$

This establishes the claim. □

Now suppose $(\overline{y}_i, \overline{y}_j)$ are equilibrium budget choices.

If $\frac{1}{2}V^I > V^{II} + \frac{1}{2}\alpha$, then from Claim 2, (ii) cannot hold. Since at least one of (i) or (ii) must hold, (i) must hold and then from Claim 1, $\overline{y}_j = V^I - V^{II}$.

If $\frac{1}{2}V^I < V^{II} + \frac{1}{2}\alpha$ then from Claim 1, (i) cannot hold. Again, since at least one of (i) or (ii) must hold, (ii) must hold and then from Claim 2, $y_j = \frac{1}{2}\left(V^I + \alpha\right)$.

If $\frac{1}{2}V^I = V^{II} + \frac{1}{2}\alpha$ then either (i) or (ii) may hold. In either case, Claims 1 and 2 imply that $\overline{y}_j = V^I - V^{II} = \frac{1}{2}\left(V^I + \alpha\right)$.

The prices are then determined by Lemma 4 and this implies that the payoffs are unique. ■

A.4.2 Complements

Lemma 6 *Suppose* $\alpha > 0$. *The budgets* $(\overline{y}_i, \overline{y}_j)$ *and the prices* $(\overline{p}^I, \overline{p}^{II})$ *constitute an equilibrium outcome of the game with endogenous budgets if and only if:*

$$\begin{array}{lll} \overline{y}_i \geq V^I + 2\alpha & \overline{p}^I = V^I - V^{II} + \alpha & \textit{if } \frac{1}{2}V^I \geq V^{II} - \frac{1}{2}\alpha \\ \overline{y}_j = V^I - V^{II} + \alpha & \overline{p}^{II} = 0 & \end{array}$$

$$\begin{array}{lll} \overline{y}_i \geq \frac{1}{2}V^I + V^{II} + \frac{3}{2}\alpha & \overline{p}^I = \frac{1}{2}\left(V^I + \alpha\right) & \textit{if } \frac{1}{2}V^I < V^{II} - \frac{1}{2}\alpha \\ \overline{y}_j = \frac{1}{2}\left(V^I + \alpha\right) & \overline{p}^{II} = 0 & \end{array}$$

All equilibria result in the same payoffs (except for a relabelling of the bidders).

The formal proof of Lemma 6 is similar to that of Lemma 5 and is omitted for the sake of brevity. It is available from the authors.

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