



Modeling Transformations

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Princeton University
COS 426, Spring 2006



Modeling Transformations

- Specify transformations for objects
 - Allows definitions of objects in own coordinate systems
 - Allows use of object definition multiple times in a scene



Mike Marr, COS 426

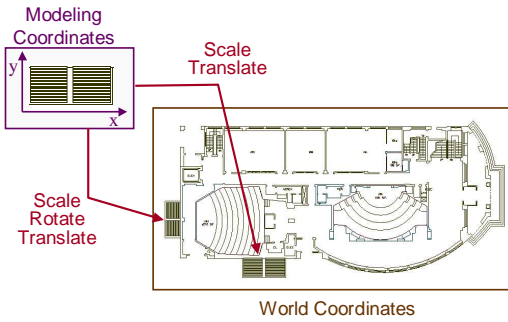


Overview

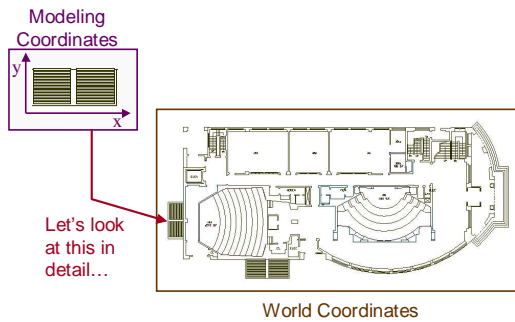
- 2D Transformations
 - Basic 2D transformations
 - Matrix representation
 - Matrix composition
- 3D Transformations
 - Basic 3D transformations
 - Same as 2D
- Transformation Hierarchies
 - Scene graphs
 - Ray casting



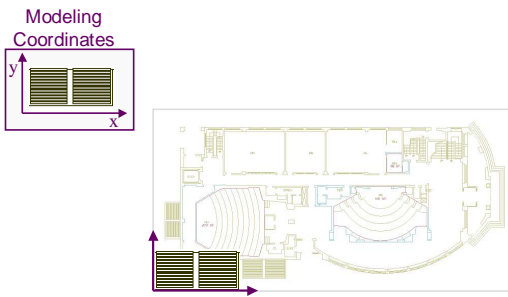
2D Modeling Transformations



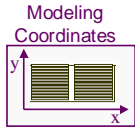
2D Modeling Transformations



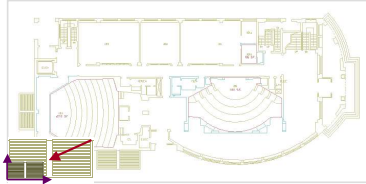
2D Modeling Transformations



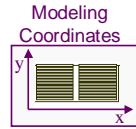
2D Modeling Transformations



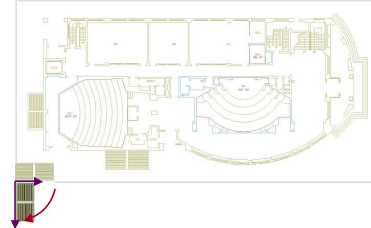
Scale .3, .3



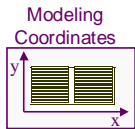
2D Modeling Transformations



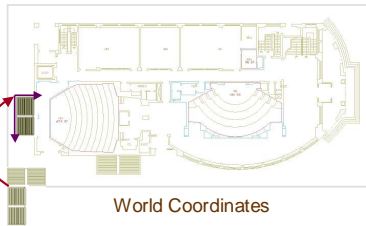
Scale .3, .3
Rotate -90



2D Modeling Transformations



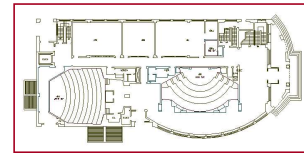
Scale .3, .3
Rotate -90
Translate 5, 3



Basic 2D Transformations



- Translation:
 - $x' = x + tx$
 - $y' = y + ty$
- Scale:
 - $x' = x * sx$
 - $y' = y * sy$
- Shear:
 - $x' = x + hx*y$
 - $y' = y + hy*x$
- Rotation:
 - $x' = x*cos\theta - y*sin\theta$
 - $y' = x*sin\theta + y*cos\theta$

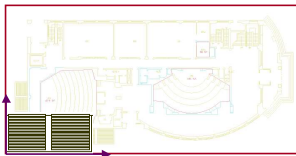


Transformations can be combined (with simple algebra)

Basic 2D Transformations



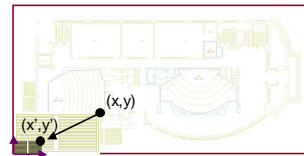
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Basic 2D Transformations



- Translation:
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 - $y' = x*sin\theta + y*cos\theta$



$$\begin{matrix} x' = x * sx \\ y' = y * sy \end{matrix}$$

Basic 2D Transformations



- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

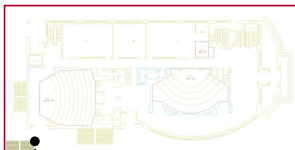
- $x' = x * sx$
- $y' = y * sy$

- Shear:

- $x' = x + hx*y$
- $y' = y + hy*x$

- Rotation:

- $x' = x*cos\theta - y*sin\theta$
- $y' = x*sin\theta + y*cos\theta$



$$\begin{aligned} x' &= (x*sx)*cos\theta - (y*sy)*sin\theta \\ y' &= (x*sx)*sin\theta + (y*sy)*cos\theta \end{aligned}$$

Basic 2D Transformations



- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

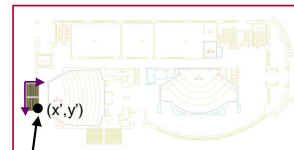
- $x' = x * sx$
- $y' = y * sy$

- Shear:

- $x' = x + hx*y$
- $y' = y + hy*x$

- Rotation:

- $x' = x*cos\theta - y*sin\theta$
- $y' = x*sin\theta + y*cos\theta$



$$\begin{aligned} x' &= ((x*sx)*cos\theta - (y*sy)*sin\theta) + tx \\ y' &= ((x*sx)*sin\theta + (y*sy)*cos\theta) + ty \end{aligned}$$

Basic 2D Transformations



- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

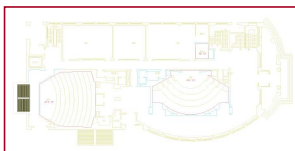
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- Rotation:

- $x' = x*cos\theta - y*sin\theta$
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$$\begin{aligned} x' &= ((x*sx)*cos\theta - (y*sy)*sin\theta) + tx \\ y' &= ((x*sx)*sin\theta + (y*sy)*cos\theta) + ty \end{aligned}$$

Overview



- 2D Transformations

- Basic 2D transformations
- Matrix representation
- Matrix composition

- 3D Transformations

- Basic 3D transformations
- Same as 2D

- Transformation Hierarchies

- Scene graphs
- Ray casting

Matrix Representation



- Represent 2D transformation by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Multiply matrix by column vector

↔ apply transformation to point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

Matrix Representation



- Transformations combined by multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrices are a convenient and efficient way to represent a sequence of transformations!

2x2 Matrices



- What types of transformations can be represented with a 2x2 matrix?

2D Identity?

$$\begin{aligned} x' &= x \\ y' &= y \end{aligned} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Scale around (0,0)?

$$\begin{aligned} x' &= sx * x \\ y' &= sy * y \end{aligned} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} sx & 0 \\ 0 & sy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices



- What types of transformations can be represented with a 2x2 matrix?

2D Rotate around (0,0)?

$$\begin{aligned} x' &= \cos \Theta * x - \sin \Theta * y \\ y' &= \sin \Theta * x + \cos \Theta * y \end{aligned} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Shear?

$$\begin{aligned} x' &= x + shx * y \\ y' &= shy * x + y \end{aligned} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & shx \\ shy & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices



- What types of transformations can be represented with a 2x2 matrix?

2D Mirror over Y axis?

$$\begin{aligned} x' &= -x \\ y' &= y \end{aligned} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Mirror over (0,0)?

$$\begin{aligned} x' &= -x \\ y' &= -y \end{aligned} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices



- What types of transformations can be represented with a 2x2 matrix?

2D Translation?

$$\begin{aligned} x' &= x + tx \\ y' &= y + ty \end{aligned} \quad \text{NO!}$$

Only linear 2D transformations can be represented with a 2x2 matrix

Linear Transformations



- Linear transformations are combinations of ...
 - Scale,
 - Rotation,
 - Shear, and
 - Mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Properties of linear transformations:

- Satisfies: $T(s_1\mathbf{p}_1 + s_2\mathbf{p}_2) = s_1T(\mathbf{p}_1) + s_2T(\mathbf{p}_2)$
- Origin maps to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition

2D Translation



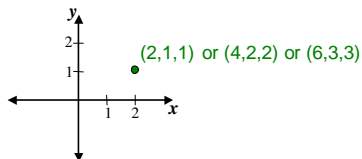
- 2D translation represented by a 3x3 matrix
 - Point represented with *homogeneous coordinates*

$$\begin{aligned} x' &= x + tx \\ y' &= y + ty \end{aligned} \quad \begin{matrix} \downarrow & & \downarrow \\ \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \end{matrix}$$

Homogeneous Coordinates



- Add a 3rd coordinate to every 2D point
 - (x, y, w) represents a point at location $(x/w, y/w)$
 - $(x, y, 0)$ represents a point at infinity
 - $(0, 0, 0)$ is not allowed



Convenient coordinate system to represent many useful transformations

Basic 2D Transformations



- Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 \\ \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & shx & 0 \\ shy & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

Affine Transformations



- Affine transformations are combinations of ...
 - Linear transformations, and
 - Translations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of affine transformations:
 - Origin does not necessarily map to origin
 - Lines map to lines
 - Parallel lines remain parallel
 - Ratios are preserved
 - Closed under composition

Projective Transformations



- Projective transformations ...
 - Affine transformations, and
 - Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of projective transformations:
 - Origin does not necessarily map to origin
 - Lines map to lines
 - Parallel lines do not necessarily remain parallel
 - Ratios are not preserved (but "cross-ratios" are)
 - Closed under composition

Overview



- 2D Transformations
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Matrix Composition



- Transformations can be combined by matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 \\ \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\mathbf{p}' = \mathbf{T}(tx, ty) \quad \mathbf{R}(\Theta) \quad \mathbf{S}(sx, sy) \quad \mathbf{p}$$



Matrix Composition



- Matrices are a convenient and efficient way to represent a sequence of transformations
 - General purpose representation
 - Hardware matrix multiply
 - Efficiency with premultiplication
 - » Matrix multiplication is associative

$$\begin{aligned} \mathbf{p}' &= (\mathbf{T} * (\mathbf{R} * (\mathbf{S} * \mathbf{p}))) \\ \mathbf{p}' &= (\mathbf{T} * \mathbf{R} * \mathbf{S}) * \mathbf{p} \end{aligned}$$



Matrix Composition



- Be aware: order of transformations matters
 - » Matrix multiplication is not commutative

$$\mathbf{p}' = \mathbf{T} * \mathbf{R} * \mathbf{S} * \mathbf{p}$$

←
→

“Global”
“Local”



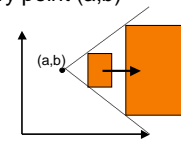
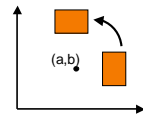
Matrix Composition



- Rotate by Θ around arbitrary point (a,b)
 - $\mathbf{M} = \mathbf{T}(a,b) * \mathbf{R}(\Theta) * \mathbf{T}(-a,-b)$

The trick:
 First, translate (a,b) to the origin.
 Next, do the rotation about origin.
 Finally, translate back.
- Scale by s_x, s_y around arbitrary point (a,b)
 - $\mathbf{M} = \mathbf{T}(a,b) * \mathbf{S}(s_x, s_y) * \mathbf{T}(-a,-b)$

(Use the same trick.)



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3D Transformations



- Same idea as 2D transformations
 - Homogeneous coordinates: (x,y,z,w)
 - 4x4 transformation matrices

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Basic 3D Transformations



$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \text{Identity}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \text{Scale}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & tx \\ 0 & 1 & 0 & ty \\ 0 & 0 & 1 & tz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \text{Translation}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \text{Mirror over X axis}$$

Basic 3D Transformations



Rotate around Z axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 & 0 \\ \sin \Theta & \cos \Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around Y axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos \Theta & 0 & -\sin \Theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin \Theta & 0 & \cos \Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around X axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \Theta & -\sin \Theta & 0 \\ 0 & \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Overview

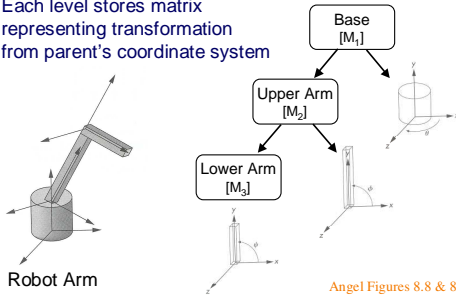


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Transformation Hierarchies



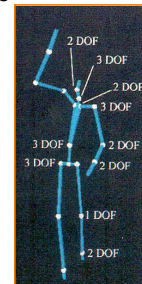
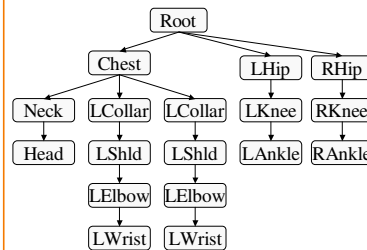
- Scene may have hierarchy of coordinate systems
 - Each level stores matrix representing transformation from parent's coordinate system



Transformation Example 1



- Well-suited for humanoid characters



Transformation Example 1

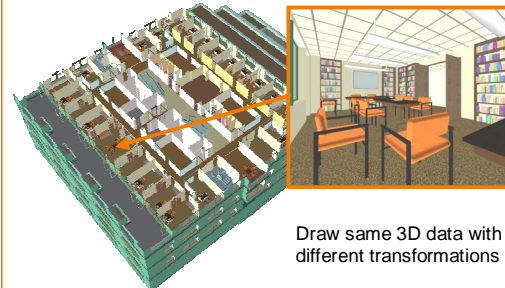


Mike Marr, COS 426,
Princeton University, 1995

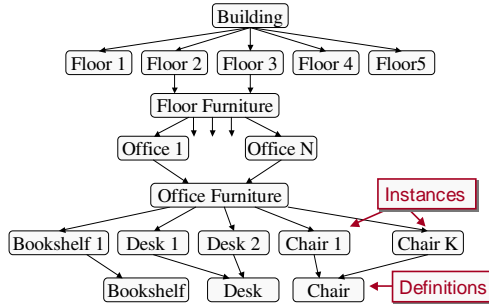
Transformation Example 2



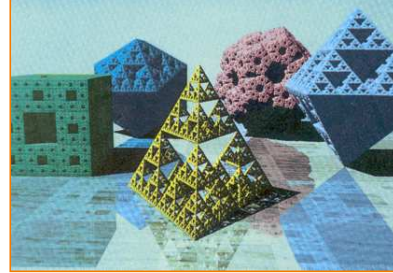
- An object may appear in a scene multiple times



Transformation Example 2

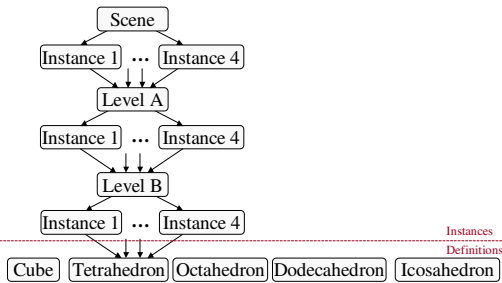


Transformation Example 3



H&B Figure 109

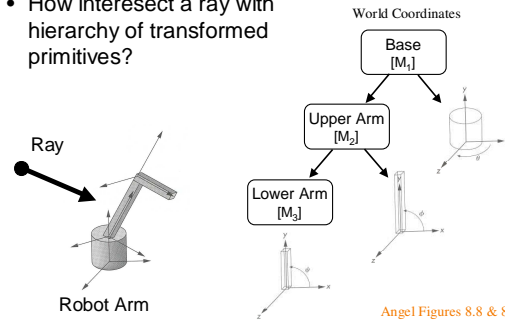
Transformation Example 3



Ray Casting With Hierarchies



- How intersect a ray with hierarchy of transformed primitives?

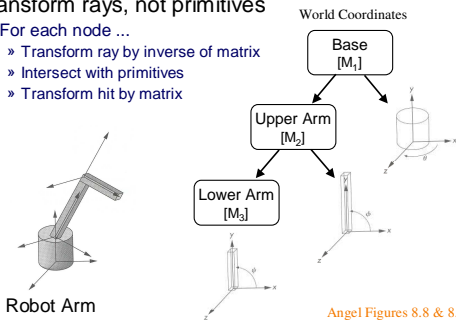


Angel Figures 8.8 & 8.9

Ray Casting With Hierarchies



- Transform rays, not primitives
 - For each node ...
 - » Transform ray by inverse of matrix
 - » Intersect with primitives
 - » Transform hit by matrix



Angel Figures 8.8 & 8.9

Summary



- Coordinate systems
 - World coordinates
 - Modeling coordinates
- Representations of 3D modeling transformations
 - 4x4 Matrices
 - » Scale, rotate, translate, shear, projections, etc.
 - » Not arbitrary warps
- Composition of 3D transformations
 - Matrix multiplication (order matters)
 - Transformation hierarchies