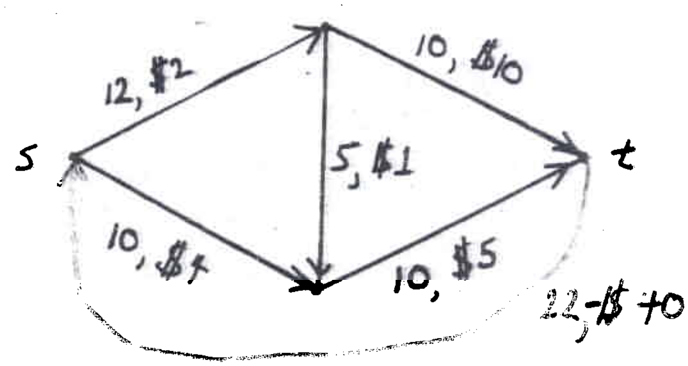
# Minimum - Cost Network Flow

In addition to a capacity, each edge has a real-valued cost per unit of flow.

A minimum-cost (maximum) flow is a maximum flow whose total cost (sum of edge flows times edge costs) is minimum.

Problem: find a minimum-cost flow in a given network.




$n$ = # vertices

$m$ = # edges

$U$ = max capacity (if integers)

$C$ = max cost (if integers)

# Two Naive Approaches

(1) Repeat: augment along a cheapest path in the residual network.

Each augmentation takes a shortest path computation.

Shortest paths can be found using Dijkstra's algorithm if costs are kept non-negative using price transformation (primal-dual method of linear programming).

Time: $O(nU(m + n\log n))$ (not polynomial)

(2) repeat $\left\{ \begin{array}{l} \text{In network of zero-cost residual edges, find a maximum flow.} \\ \text{Augment the flow and update the prices.} \end{array} \right.$

(find all paths of a given cost at once)

Time: $O(nC(nm\log(n^2/m)))$ (not polynomial)

$G = (V, E)$ symmetric directed graph

$(v,w) \in E$ iff $(w,v) \in E$

$|V| = n$, $|E| = m$. $m \geq n \geq 2$ $E(v) = \{w \mid (v,w) \in E\}$

arc capacities $u(v,w) : (v,w) \in E$

arc costs $c(v,w) : (v,w) \in E$

cost function is antisymmetric: $c(v,w) = -c(w,v)$

Circulation $f : E \to R$

$f(v,w) \leq u(v,w)$ $\forall (v,w) \in E$ capacity constraints

$f(v,w) = -f(w,v)$ $\forall (v,w) \in E$ flow antisymmetry

$\sum_{w \in E(v)} f(v,w) = 0$ $\forall v \in V$ flow conservation

Cost of $f$: $\frac{1}{2} \sum_{(v,w) \in E} f(v,w)\, c(v,w)$

Reformulated problem: Find a circulation of minimum cost: add a return arc from $t$ to $s$ of infinite capacity and large negative cost ($-nC$).

residual capacity $u_f(v,w) = u(v,w) - f(v,w)$ or $f(w,v)$

residual arc $(v,w)$: $u_f(v,w) > 0$

residual cycle: a (simple) cycle of residual arcs

<u>length</u> of cycle = number of arcs, $\ell(\Gamma)$

<u>cost</u> of cycle = sum of arc costs = $c(\Gamma)$

<u>mean cost</u> of cycle = $c(\Gamma)/\ell(\Gamma)$

<u>negative</u> cycle: $c(\Gamma) < 0$

## Naive Approach

(1) Repeat: augment along a cheapest path in the residual graph.

Each augmentation takes a shortest path computation.

Shortest paths can be found using Dijkstra's algorithm if costs are kept non-negative using "prices" to transform costs (primal-dual method).

Time: $O(nU(m + n\log n))$ (not polynomial)

(2) repeat $\begin{cases} \text{In network of zero-cost edges find a maximum flow.} \\ \text{Augment the flow and update the prices.} \end{cases}$

Time: $O(nC(nm\log(n^2/m)))$ (not polynomial)

<u>Theorem</u> (Busacker and Saaty, 1965): A circulation $f$ is minimum-cost iff there is no negative residual cycle.

<u>Algorithm</u> (Klein, 1967)

0. Find any circulation $f$ (by a max flow computation)

1. While $\exists$ negative cycle $\Gamma$, <u>cancel</u> $\Gamma$ by increasing $f$ on all arcs of $\Gamma$ by $\min\{u_f(v,w) : (v,w) \in \Gamma\}$.

\# iterations can be exponential (or infinite)

How to choose cycles for canceling to minimize #iterations, running time?

minimum cost?

minimum length?

maximum length?

maximum capacity?

maximum cost decrease?

## Primal Network Simplex Algorithm: Definitions

~~[illegible]~~

$\{v,w\}$ is residual if both $(v,w)$ and $(w,v)$ are residual

$f$ is basic if the set of residual edges forms a forest

The algorithm maintains a basic circulation $f$ and a basis tree $T$ such that $T$ contains every residual edge.

Any nontree arc $(v,w)$ defines a basic cycle $\Gamma_T(v,w)$ consisting of $(v,w)$ and the path of tree arcs from $w$ to $v$.

(We regard each tree edge as consisting of a pair of tree arcs.)

An arc $(v,w)$ is pseudoresidual if it is residual or a tree arc.

A simple cycle is pseudoresidual if it consists only of pseudoresidual arcs.

# Our Results

Minimum-mean cycle canceling: Always cancel a cycle of minimum mean cost.

Theorem: # cancellations $= O(nm^2 \log n)$. If costs are integers of maximum magnitude $C$, # cancellations $= O(nm \log(nC))$.

Time to find a minimum mean cycle $= O(nm)$ (Karp, 1978)

A variant of this approach gives a "practical" algorithm with a running time of $O(nm \log n \min\{\log(nC), m \log n\})$.

# Price Function (Dual Variables)

$p: V \to \mathbb{R}$ reduced arc cost $c_p(v,w) = c(v,w) + p(v) - p(w)$

<u>Theorem</u> (Ford and Fulkerson, 1962): A circulation $f$ is minimum-cost iff $\exists p$ such that, $\forall (v,w) \in E$,

$u_f(v,w) > 0$ implies $c_p(v,w) \geq 0$.

## $\epsilon$-optimality

For $\epsilon > 0$, a circulation $f$ is $\epsilon$-optimal with respect to a price function $p$ iff, $\forall (v,w) \in E$,

$u_f(v,w) > 0$ implies $c_p(v,w) \geq -\epsilon$.

$\epsilon(f)$ = minimum $\epsilon \geq 0$ for which $f$ is $\epsilon$-optimal with respect to some $p$.

<u>Theorem</u> (Bertsekas, 1986): If costs are integral and $\epsilon < 1/n$, any $\epsilon$-optimal circulation is minimum-cost.

Idea: minimum mean cycle canceling reduces $\epsilon(f)$ by a measurable amount, after enough cancellations.

Note: The cost of a cycle is the same as its reduced cost.

Key question: What is $\epsilon(f)$?

Let $\mu(f)$ be the mean cost of a minimum mean residual cycle with respect to circulation $f$.

<u>Theorem</u>: $\epsilon(f) = \max\{0, -\mu(f)\}$.

<u>Proof</u>: Use properties of shortest paths, e.g. shortest paths exist iff there are no negative cycles.

# Analysis of Minimum Mean Cycle Canceling

__Lemma__: Canceling a minimum mean cycle cannot increase $\epsilon(f)$.

__Lemma__: After $m$ cancellations, $\epsilon(f)$ has decreased by a factor of $(n-1)/n$ or better.

__Theorem__: # cancellations $= O(nm \log(nC))$.

__Lemma__: If $f$ and $f'$ are both $\epsilon$-optimal and $|c_p(v,w)| > 2n\epsilon$, where $f$ is $\epsilon$-optimal with respect to $p$, then $f(v,w) = f'(v,w)$.

__Theorem__: # cancellations $= O(nm^2 \log n)$.

# A "Practical" Variant

Maintain a price function $p$ along with a circulation $f$.

Call an arc $(v,w)$ <u>eligible</u> if $u_f(v,w)>0$ and $c_p(v,w)<0$.

Let $\epsilon(f,p) = -\min\left(\{c_p(v,w) \mid u_f(v,w)>0\} \cup \{0\}\right)$.

## Algorithm

0. Let $f$ be any circulation and let $p=0$.

1. Repeat until $\epsilon(f,p) < 1/n$:

   a. Find an eligible cycle and cancel it.

   b. If the subgraph of eligible arcs is acyclic, modify $p$ to reduce $\epsilon(f,p)$ by a factor of at least $(n-1)/n$.

# Analysis

There are at most $m$ iterations of 1a between iterations of 1b.

All iterations of 1a between two iterations of 1b take a total of $O(m \log n)$ time using dynamic trees.

One iteration of 1b takes $O(m)$ time.

$O(n)$ iterations of 1b reduce $\epsilon(f,p)$ by a constant factor.

$\therefore$ $O(nm \log n \log(nC))$ total time.

If every $n^{th}$ iteration of 1b reduces $\epsilon(f,p)$ as much as possible, then the amortized time per iteration of 1b is still $O(m)$ (every $n^{th}$ takes $O(nm)$).

$\therefore$ $O(nm^2 (\log n)^2)$ total time.

# Polynomial-Time Algorithms

| Date | Discoverer | Time |
|---|---|---|
| 1972 | Edmonds and Karp | $O(m(\log U)(m + n\log n))$ |
| 1980 | Rock | $O(m(\log U)(m + n\log n))$ |
| 1980 | Rock | $O(n(\log C)\ nm\log(n^2/m))$ |
| 1984 | Tardos | $O(m^4)$ |
| 1984 | Orlin | $O(m^2(\log n)(m + n\log n))$ |
| 1985 | Fujishige | $O(m^3\log n)$ |
| 1985 | Bland and Jensen | $O(n(\log C)\ nm\log(n^2/m))$ |
| 1986 | Galil and Tardos | $O(n^2(\log n)(m + n\log n))$ |
| 1987 | Goldberg and Tarjan | $O((\log nC)\min\{n^3, nm\log n\})$ |