13. Randomized Algorithms

Randomization

Algorithmic design patterns.
- Greed.
- Divide-and-conquer.
- Dynamic programming.
- Network flow.
- Randomization.

in practice, access to a pseudo-random number generator

Randomization. Allow fair coin flip in unit time.

Why randomize? Can lead to simplest, fastest, or only known algorithm for a particular problem.

Ex. Symmetry breaking protocols, graph algorithms, quicksort, hashing, load balancing, Monte Carlo integration, cryptography.

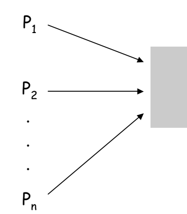
13.1 Contention Resolution

Contention Resolution in a Distributed System

Contention resolution. Given n processes $P_1, \ldots, P_n$, each competing for access to a shared database. If two or more processes access the database simultaneously, all processes are locked out. Devise protocol to ensure all processes get through on a regular basis.

Restriction. Processes can't communicate.

Challenge. Need symmetry-breaking paradigm.



Contention Resolution: Randomized Protocol

Protocol. Each process requests access to the database at time t with probability p = 1/n.

Claim. Let S[i, t] = event that process i succeeds in accessing the database at time t. Then $1/(e \cdot n) \le \Pr[S(i, t)] \le 1/(2n)$.

Pf. By independence, $\Pr[S(i, t)] = p\,(1-p)^{n-1}$.

process i requests access — none of remaining n-1 processes request access

- Setting p = 1/n, we have $\Pr[S(i, t)] = 1/n\ \underbrace{(1 - 1/n)^{n-1}}_{\text{between } 1/e \text{ and } 1/2}$. ▪

value that maximizes Pr[S(i, t)]

Useful fact. As n increases from 2, the function:
- $(1 - 1/n)^n$ converges monotonically from 1/4 up to 1/e
- $(1 - 1/n)^{n-1}$ converges monotonically from 1/2 down to 1/e.

Contention Resolution: Randomized Protocol

Claim. The probability that process i fails to access the database in en rounds is at most 1/e. After $e \cdot n(c \ln n)$ rounds, the probability is at most n^{-c}.

Pf. Let F[i, t] = event that process i fails to access database in rounds 1 through t. By independence and previous claim, we have $\Pr[F(i, t)] \le (1 - 1/(en))^{t}$.

- Choose $t = \lceil e \cdot n \rceil$: $$\Pr[F(i,t)] \le \left(1-\tfrac{1}{en}\right)^{\lceil en \rceil} \le \left(1-\tfrac{1}{en}\right)^{en} \le \tfrac{1}{e}$$
- Choose $t = \lceil e \cdot n \rceil \lceil c \ln n \rceil$: $$\Pr[F(i,t)] \le \left(\tfrac{1}{e}\right)^{c \ln n} = n^{-c}$$

Contention Resolution: Randomized Protocol

Claim. The probability that all processes succeed within $2e \cdot n \ln n$ rounds is at least 1 - 1/n.

Pf. Let F[t] = event that at least one of the n processes fails to access database in any of the rounds 1 through t.

$$\Pr\big[F[t]\big] = \Pr\left[\bigcup_{i=1}^{n} F[i,t]\right] \underset{\text{union bound}}{\le} \sum_{i=1}^{n} \Pr[F[i,t]] \underset{\text{previous slide}}{\le} n\left(1-\tfrac{1}{en}\right)^{t}$$

- Choosing $t = 2 \lceil en \rceil \lceil c \ln n \rceil$ yields $\Pr[F[t]] \le n \cdot n^{-2} = 1/n$. ▪

Union bound. Given events $E_1, \ldots, E_n$, $\Pr\left[\bigcup_{i=1}^{n} E_i\right] \le \sum_{i=1}^{n} \Pr[E_i]$

13.2 Global Minimum Cut

Global Minimum Cut

Global min cut. Given an undirected graph $G = (V, E)$ find a cut (A, B) of minimum cardinality.

Applications. Partitioning items in a database, identify clusters of related documents, network reliability, network design, circuit design, TSP solvers.

Network flow solution.

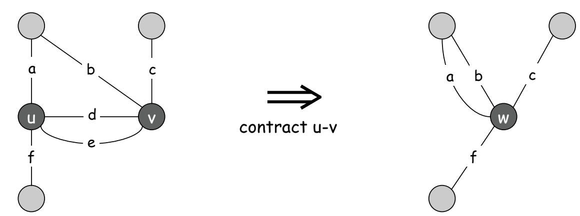
- Replace every edge (u, v) with two antiparallel edges (u, v) and (v, u).
- Pick some vertex s and compute min s-v cut separating s from each other vertex $v \in V$.

False intuition. Global min-cut is harder than min s-t cut.

Contraction Algorithm

Contraction algorithm. [Karger 1995]

- Pick an edge $e = (u, v)$ uniformly at random.
- Contract edge e.
  - replace u and v by single new super-node w
  - preserve edges, updating endpoints of u and v to w
  - keep parallel edges, but delete self-loops
- Repeat until graph has just two nodes v_1 and v_2.
- Return the cut (all nodes that were contracted to form v_1).

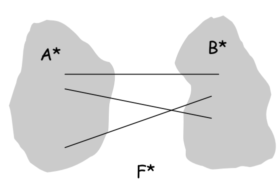


Contraction Algorithm

Claim. The contraction algorithm returns a min cut with prob $\geq 2/n^2$.

Pf. Consider a global min-cut (A^*, B^*) of G. Let F^* be edges with one endpoint in A^* and the other in B^*. Let $k = |F^*|$ = size of min cut.

- In first step, algorithm contracts an edge in F^* probability $k / |E|$.
- Every node has degree $\geq k$ since otherwise (A^*, B^*) would not be min-cut. $\Rightarrow$ $|E| \geq \frac{1}{2}kn$.
- Thus, algorithm contracts an edge in F^* with probability $\leq 2/n$.



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- Let G' be graph after j iterations. There are $n' = n-j$ supernodes.
- Suppose no edge in F^* has been contracted. The min-cut in G' is still k.
- Since value of min-cut is k, $|E'| \geq \frac{1}{2}kn'$.
- Thus, algorithm contracts an edge in F^* with probability $\leq 2/n'$.

- Let E_j = event that an edge in F^* is not contracted in iteration j.

$$
\begin{aligned}
\Pr[E_1 \cap E_2 \cdots \cap E_{n-2}] &= \Pr[E_1] \times \Pr[E_2 \mid E_1] \times \cdots \times \Pr[E_{n-2} \mid E_1 \cap E_2 \cdots \cap E_{n-3}] \\
&\geq \left(1-\tfrac{2}{n}\right)\left(1-\tfrac{2}{n-1}\right)\cdots\left(1-\tfrac{2}{4}\right)\left(1-\tfrac{2}{3}\right) \\
&= \left(\tfrac{n-2}{n}\right)\left(\tfrac{n-3}{n-1}\right)\cdots\left(\tfrac{2}{4}\right)\left(\tfrac{1}{3}\right) \\
&= \tfrac{2}{n(n-1)} \\
&\geq \tfrac{2}{n^2}
\end{aligned}
$$

Contraction Algorithm

Amplification. To amplify the probability of success, run the contraction algorithm many times.

Claim. If we repeat the contraction algorithm $n^2 \ln n$ times with independent random choices, the probability of failing to find the global min-cut is at most $1/n^2$.

Pf. By independence, the probability of failure is at most

$$\left(1-\frac{2}{n^2}\right)^{n^2 \ln n} = \left[\left(1-\frac{2}{n^2}\right)^{\frac{1}{2}n^2}\right]^{2\ln n} \underset{\uparrow}{\le} \left(e^{-1}\right)^{2\ln n} = \frac{1}{n^2}$$

$(1 - 1/x)^x \le 1/e$

Global Min Cut: Context

Remark. Overall running time is slow since we perform $\Theta(n^2 \log n)$ iterations and each takes $\Omega(m)$ time.

Improvement. [Karger-Stein 1996] $O(n^2 \log^3 n)$.

- Early iterations are less risky than later ones: probability of contracting an edge in min cut hits 50% when $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm until $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm twice on resulting graph, and return best of two cuts.

Extensions. Naturally generalizes to handles positive weights.

Best known. [Karger 2000] $O(m \log^3 n)$.

faster than best known max flow algorithm or deterministic global min cut algorithm

13.3 Linearity of Expectation

Expectation

Expectation. Given a discrete random variables X, its expectation E[X] is defined by:

$$E[X] = \sum_{j=0}^{\infty} j \Pr[X = j]$$

Linearity of expectation. Given two random variables X and Y defined over the same probability space, E[X + Y] = E[X] + E[Y].

Waiting for a first success. Coin is heads with probability p and tails with probability 1-p. How many independent flips X until first heads?

$$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X = j] = \sum_{j=0}^{\infty} j\, \underbrace{(1-p)^{j-1}}_{\text{j-1 tails}} \underbrace{p}_{\text{1 head}} = \frac{p}{1-p} \sum_{j=0}^{\infty} j\,(1-p)^j = \frac{p}{1-p} \cdot \frac{1-p}{p^2} = \frac{1}{p}$$

Guessing Cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Memoryless guessing. No psychic abilities; can't even remember what's been turned over already. Guess a card from full deck uniformly at random.

Claim. The expected number of correct guesses is 1.
Pf.
- Let X_i = 1 if i^{th} prediction is correct and 0 otherwise.
- Let X = number of correct guesses = $X_1 + \ldots + X_n$.
- $E[X_i] = 0 \cdot Pr[X_i = 0] + 1 \cdot Pr[X_i = 1] = Pr[X_i = 1] = 1/n$.
- $E[X] = E[X_1] + \ldots + E[X_n] = 1$. ▪

Guessing Cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Guessing with memory. Guess a card uniformly at random from cards not yet seen.

Claim. The expected number of correct guesses is $\Theta(\log n)$.
Pf.
- Let X_i = 1 if i^{th} prediction is correct and 0 otherwise.
- Let X = number of correct guesses = $X_1 + \ldots + X_n$.
- $E[X_i] = 1 / (n - i - 1)$.
- $E[X] = 1/n + \ldots + 1/2 + 1/1 = H(n)$. ▪

$\ln(n+1) < H(n) < 1 + \ln n$

Coupon Collector

Coupon collector. Each box of cereal contains a coupon. There are n different types of coupons. Assuming all boxes are equally likely to contain each coupon, how many boxes before you have ≥ 1 coupon of each type?

Claim. The expected number of steps is $\Theta(n \log n)$.
Pf.
- Phase j = time between j and j+1 distinct coupons.
- Let X_j = number of steps you spend in phase j.
- Let X = number of steps in total = $X_0 + X_1 + \ldots + X_{n-1}$.

$$E[X] = \sum_{j=0}^{n-1} E[X_j] = \sum_{j=0}^{n-1} \frac{n}{n-j} = n \sum_{i=1}^{n} \frac{1}{i} = n H(n)$$

prob of success = (n-j)/n
⇒ expected waiting time = n/(n-j)

13.4 MAX 3-SAT

Maximum 3-Satisfiability

exactly 3 distinct literals per clause

MAX-3SAT. Given 3-SAT formula, find a truth assignment that satisfies as many clauses as possible.

$$\begin{aligned}
C_1 &= x_2 \vee \overline{x_3} \vee \overline{x_4} \\
C_2 &= x_2 \vee x_3 \vee \overline{x_4} \\
C_3 &= \overline{x_1} \vee x_2 \vee x_4 \\
C_4 &= \overline{x_1} \vee \overline{x_2} \vee x_3 \\
C_5 &= x_1 \vee \overline{x_2} \vee \overline{x_4}
\end{aligned}$$

Remark. NP-hard search problem.

Simple idea. Flip a coin, and set each variable true with probability $\frac{1}{2}$, independently for each variable.

Maximum 3-Satisfiability: Analysis

Claim. Given a 3-SAT formula with k clauses, the expected number of clauses satisfied by a random assignment is 7k/8.

Pf. Consider random variable $Z_j = \begin{cases} 1 & \text{if clause } C_j \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases}$

- Let Z = weight of clauses satisfied by assignment Z_j.

linearity of expectation

$$\begin{aligned}
E[Z] &= \sum_{j=1}^{k} E[Z_j] \\
&= \sum_{j=1}^{k} \Pr[\text{clause } C_j \text{ is satisfied}] \\
&= \tfrac{7}{8}k
\end{aligned}$$

Corollary. For any instance of 3-SAT, there exists a truth assignment that satisfies at least a 7/8 fraction of all clauses.

Maximum 3-Satisfiability: Analysis

Q. Can we turn this idea into a 7/8-approximation algorithm? In general, a random variable can almost always be below its mean.

Lemma. The probability that a random assignment satisfies ≥ 7k/8 clauses is at least 1/(8k).

Pf. Let p_j be probability that exactly j clauses are satisfied; let p be probability that ≥ 7k/8 clauses are satisfied.

$$\begin{aligned}
\tfrac{7}{8}k = E[Z] &= \sum_{j \geq 0} j\, p_j \\
&= \sum_{j < 7k/8} j\, p_j + \sum_{j \geq 7k/8} j\, p_j \\
&\leq \left(\tfrac{7k}{8} - \tfrac{1}{8}\right) \sum_{j < 7k/8} p_j + k \sum_{j \geq 7k/8} p_j \\
&\leq \left(\tfrac{7}{8}k - \tfrac{1}{8}\right) \cdot 1 + k\, p
\end{aligned}$$

Rearranging terms yields p ≥ 1 / (8k). ▪

Maximum 3-Satisfiability: Analysis

Johnson's algorithm. Repeatedly generate random truth assignments until one of them satisfies ≥ 7k/8 clauses.

Theorem. Johnson's algorithm is a 7/8-approximation algorithm.

Pf. By previous lemma, each iteration succeeds with probability at least 1/(8k). By the waiting-time bound, the expected number of trials to find the satisfying assignment is at most 8k. ▪

Maximum Satisfiability

Extensions.

- Allow one, two, or more literals per clause.
- Find max weighted set of satisfied clauses.

Theorem. [Asano-Williamson 2000] There exists a 0.784-approximation algorithm for MAX-SAT.

Theorem. [Karloff-Zwick 1997, Zwick+computer 2002] There exists a 7/8-approximation algorithm for version of MAX-3SAT where each clause has at most 3 literals.

Theorem. [Håstad 1997] Unless P = NP, no ρ-approximation algorithm for MAX-3SAT (and hence MAX-SAT) for any ρ > 7/8.

very unlikely to improve over simple randomized algorithm for MAX-3SAT

Monte Carlo vs. Las Vegas Algorithms

Monte Carlo algorithms. Guaranteed to run in poly-time, likely to find correct answer.
Ex: Contraction algorithm for global min cut.

Las Vegas algorithms. Guaranteed to find correct answer, likely to run in poly-time.
Ex: Randomized quicksort, Johnson's MAX-3SAT algorithm.

stop algorithm after a certain point

Remark. Can always convert a Las Vegas algorithm into Monte Carlo, but no known method to convert the other way.

RP and ZPP

RP. [Monte Carlo] Decision problems solvable with one-sided error in poly-time.

One-sided error.

Can decrease probability of false negative to 2^{-100} by 100 independent repetitions

- If the correct answer is `no`, always return `no`.
- If the correct answer is `yes`, return `yes` with probability $\geq \frac{1}{2}$.

ZPP. [Las Vegas] Decision problems solvable in expected poly-time.

running time can be unbounded, but on average it is fast

Theorem. P $\subseteq$ ZPP $\subseteq$ RP $\subseteq$ NP.

Fundamental open questions. To what extent does randomization help? Does P = ZPP? Does ZPP = RP? Does RP = NP?

13.6 Universal Hashing

Dictionary Data Type

Dictionary. Given a universe U of possible elements, maintain a subset $S \subseteq U$ so that inserting, deleting, and searching in S is efficient.

Dictionary interface.
- `Create()`: Initialize a dictionary with $S = \phi$.
- `Insert(u)`: Add element $u \in U$ to S.
- `Delete(u)`: Delete u from S, if u is currently in S.
- `Lookup(u)`: Determine whether u is in S.

Challenge. Universe U can be extremely large so defining an array of size |U| is infeasible.

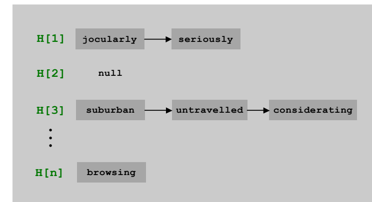
Applications. File systems, databases, Google, compilers, checksums P2P networks, associative arrays, cryptography, web caching, etc.

Hashing

Hash function. $h : U \rightarrow \{ 0, 1, \ldots, n-1 \}$.

Hashing. Create an array H of size n. When processing element u, access array element H[h(u)].

Collision. When h(u) = h(v) but $u \neq v$.
- Birthday paradox $\Rightarrow$ expect a collision after $\sqrt{n}$ random insertions.
- Separate chaining: H[i] stores linked list of elements u with h(u) = i.



Algorithmic Complexity Attacks

Deterministic hashing. If $|U| \geq n^2$, then for any fixed hash function h, there is a subset $S \subseteq U$ of n elements that all hash to same slot. Thus, $\Theta(n)$ time per search in worst-case.

When do we need stronger performance guarantees?
- Obvious situations: aircraft control, nuclear reactors.
- Surprising situations: denial-of-service attacks.

malicious adversary chooses which strings to insert into *your* hash table

Real world exploits. [Crosby-Wallach 2003]
- Bro server: send carefully chosen packets to DOS the server, using less bandwidth than a dial-up modem
- Perl 5.8.0: insert carefully chosen strings into associative array.
- Linux 2.4.20 kernel: save files with carefully chosen names.

Hashing Performance

Idealistic hash function. Maps m elements uniformly at random to n hash slots.
- Running time depends on length of chains.
- Average length of chain = α = m / n.
- Choose $n \approx m$ $\Rightarrow$ on average O(1) per insert, lookup, or delete.
- Max length of chain = $\Theta(\log m / \log \log m)$, assuming α is a constant.

see 13.10

Challenge. Find an efficiently computable hash function h that has idealized properties.

Approach. Use randomization in the choice of h.

Universal Hashing

Universal class of hash functions. [Carter-Wegman, 1980s]

- For any pair of elements u, v ∈ U, $\Pr_{h \in H}\left[\, h(u) = h(v) \,\right] \le 1/n$
- Can select random h efficiently. ← chosen uniformly at random
- Can compute h(u) efficiently.

Ex. U = { a, b, c, d, e, f }, n = 2.

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1

H = {h_1, h_2}
$\Pr_{h \in H}$ [h(a) = h(b)] = 1/2
$\Pr_{h \in H}$ [h(a) = h(c)] = 1
$\Pr_{h \in H}$ [h(a) = h(d)] = 0
. . .

not universal

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1
$h_3(x)$	0	0	1	0	1	1
$h_4(x)$	1	0	0	1	1	0

H = {h_1, h_2, h_3, h_4}
$\Pr_{h \in H}$ [h(a) = h(b)] = 1/2
$\Pr_{h \in H}$ [h(a) = h(c)] = 1/2
$\Pr_{h \in H}$ [h(a) = h(d)] = 1/2
$\Pr_{h \in H}$ [h(a) = h(e)] = 1/2
$\Pr_{h \in H}$ [h(a) = h(f)] = 0
. . .

universal

Universal Hashing

Universal hashing property. Let H be a universal class of hash functions; let h ∈ H be chosen uniformly at random from H; and let u ∈ U. For any subset S ⊆ U of size at most n, the expected number of items in S that collide with u is at most 1.

Pf. For any element s ∈ S, define indicator random variable X_s = 1 if h(s) = h(u) and 0 otherwise. Let X be a random variable counting the total number of collisions with u.

$$E_{h\in H}[X] \;=\; E[\textstyle\sum_{s\in S} X_s] \;\underset{\text{linearity of expectation}}{=}\; \textstyle\sum_{s\in S} E[X_s] \;\underset{X_s \text{ is a 0-1 random variable}}{=}\; \textstyle\sum_{s\in S} \Pr[X_s = 1] \;\underset{\text{universal (assumes } u \notin S)}{\le}\; \textstyle\sum_{s\in S} \frac{1}{n} \;=\; |S|\,\frac{1}{n} \;\le\; 1$$

Designing a Universal Family of Hash Functions

Modulus. Choose a prime number p between n and 2n.
↖ ok to use a table of primes

Integer encoding. Identify each element u ∈ U with a base-p integer of r digits: x = (x_1, x_2, …, x_r).

Hash function. Let A = set of all r-digit, base-p integers. For each a = (a_1, a_2, …, a_r) where 0 ≤ a_i < p, define

$$h_a(x) \;=\; \left(\sum_{i=1}^{r} a_i x_i \right) \bmod p$$

Hash function family. H = { h_a : a ∈ A }.

Designing a Universal Class of Hash Functions

Theorem. H = { h_a : a ∈ A } is a universal class of hash functions.

Pf. Let x = (x_1, x_2, …, x_r) and y = (y_1, y_2, …, y_r) be two distinct elements of U. We need to show that Pr[$h_a(x) = h_a(y)$] ≤ 1/n.

- Since x ≠ y, there exists an integer j such that $x_j \ne y_j$.
- We have $h_a(x) = h_a(y)$ iff

$$a_j \underbrace{(y_j - x_j)}_{z} \;=\; \underbrace{\sum_{i \ne j} a_i (x_i - y_i)}_{m} \bmod p$$

- Can assume a was chosen uniformly at random by first selecting all coordinates a_i where i ≠ j, then selecting a_j at random. Thus, we can assume a_i is fixed for all coordinates i ≠ j.
- Since p is prime, $a_j = m z^{-1}$ is the unique solution among p possibilities.
- Thus Pr[$h_a(x) = h_a(y)$] = 1/p ≤ 1/n. ▪

Bloom Filters

Set Membership

Set membership. Represent of a set S of m elements from universe U where |U| >> |S| to support membership queries.

Ex. ISP caches web pages, especially large data files like images and video. Client requests URL. Server needs to quickly determine whether the page is in its cache. False positive undesirable, but acceptable; don't want false negatives.

More applications.

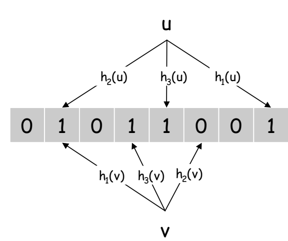
Reference: A. Broder and M. Mitzenmacher. *Network applications of bloom filters: A survey.* In Proceeding of Allerton Conference, 2002.

- Packet routing.
- Unsuitable password list.
- URLs in web proxy cache.
- Early Unix spell checkers.
- Network intrusion detection.
- Set intersection for keyword search.
- peed up semijoin operation in databases.
- Collaborating in overlay and P2P networks.

Bloom Filter

Bloom filter. [Bloom, 1978] Ingenious data structure to maximize space efficiency by allowing some false positives.

- Maintain one bit-array of size n ≈ 8 |S|.
- Choose k independent hash functions $h_1, \ldots, h_k$: $U \to \{1, \ldots, n\}$.
- `Insert(u)`: set $H[h_i(u)] = 1$ for each i = 1, …, k
- `Exists(u)`: check $H[h_i(u)] = 1$ for each i = 1, …, k
  - if any are 0, u is definitely not in set
  - if all are 1, u is probably in the set



u probably in S
v definitely not in S

Bloom Filter: Analysis

Parameters. m elements in S, n bits in table, k hash functions.

Claim. The probability of a false positive is at most $(0.62)^{n/m}$.

Pf.

assumes hash functions are idealized (or universal)

- $\Pr[H[i] = 0] = \left(1 - \frac{1}{n}\right)^{km} \approx e^{-km/n}$
- Pr[false positive for element u] = $\Pr[h_1(u) = \ldots = h_k(u) = 1]$.
  $$\approx \left(1 - e^{-km/n}\right)^k$$
- Optimal choice is k ≈ ln 2 (n / m).
- Pr[false positive for element u] $\approx (1/2)^k \leq (0.62)^{n/m}$.

Ex. n = 8m, k = 5, false positive rate ≈ 2.2%.

Bloom Filter: Delete

Set membership. Represent of a set S of m elements from universe U where |U| >> m to support membership queries.

- Insert.
- Exists.
- Delete.

Counting bloom filter. Count number of times each bit is set.

- Insert: add 1 to each of k counts.
- Delete: delete 1 from each of k counts.

Practice. Use 4 bits per hash value since probability of max value exceeding 15 < $1.37n \times 10^{-15}$.

Bloom Filter Summary

Advantage. No collisions to deal with, efficient use of space, provides privacy since no way to reconstruct S from bit array.
Disadvantage. No easy way to store info with keys.

Hashing. Use log m bits per element ⇒ vanishingly small error.
Bloom filter. Use O(1) bits per element ⇒ constant error.

> Bloom filter principle. Whenever a list or set is used and space is a consideration, a Bloom filter should be considered. When using a Bloom filter, consider the potential effects of false positives. *-Broder and Mitzenmacher*

13.9 Chernoff Bounds

Chernoff Bounds

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \geq E[X]$ and $\delta > 0$, we have

$$\Pr[X > (1+\delta)\mu] < \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$$

sum of independent 0-1 random variables is tightly centered on the mean

Pf. We apply a number of simple transformations.

- For any $t > 0$,

$$\Pr[X > (1+\delta)\mu] \underset{f(x) = e^{tX} \text{ is monotone in } x}{=} \Pr\left[e^{tX} > e^{t(1+\delta)\mu}\right] \underset{\text{Markov's inequality: } \Pr[X > a] \leq E[X]/a}{\leq} e^{-t(1+\delta)\mu} \cdot E[e^{tX}]$$

- Now $E[e^{tX}] \underset{\text{definition of } X}{=} E[e^{t\sum_i X_i}] \underset{\text{independence}}{=} \prod_i E[e^{tX_i}]$

Chernoff Bounds

Pf. (cont)

- Let $p_i = \Pr[X_i = 1]$. Then,

$$E[e^{tX_i}] = p_i e^t + (1-p_i)e^0 = 1 + p_i(e^t - 1) \underset{\text{for any } \alpha \ge 0,\ 1+\alpha \le e^\alpha}{\le} e^{p_i(e^t-1)}$$

- Combining everything:

$$\Pr[X > (1+\delta)\mu] \underset{\text{previous slide}}{\le} e^{-t(1+\delta)\mu} \prod_i E[e^{tX_i}] \underset{\text{inequality above}}{\le} e^{-t(1+\delta)\mu} \prod_i e^{p_i(e^t-1)} \underset{\sum_i p_i = E[X] \le \mu}{\le} e^{-t(1+\delta)\mu}\, e^{\mu(e^t-1)}$$

- Finally, choose $t = \ln(1 + \delta)$.

Chernoff Bounds

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \ge E[X]$ and $0 < \delta < 1$, we have

$$\Pr[X < (1-\delta)\mu] < e^{-\delta^2 \mu / 2}$$

Pf idea. Similar.

Remark. Not quite symmetric since only makes sense to consider $\delta < 1$.

13.10 Load Balancing

Load Balancing

Load balancing. System in which m jobs arrive in a stream and need to be processed immediately on n identical processors. Find an assignment that balances the workload across processors.

Centralized controller. Assign jobs in round-robin manner. Each processor receives at most $\lceil m/n \rceil$ jobs.

Decentralized controller. Assign jobs to processors uniformly at random. How likely is it that some processor is assigned "too many" jobs?

Load Balancing

Analysis.

- Let X_i = number of jobs assigned to processor i.
- Let Y_{ij} = 1 if job j assigned to processor i, and 0 otherwise.
- We have $E[Y_{ij}] = 1/n$
- Thus, $X_i = \sum_j Y_{ij}$, and $\mu = E[X_i] = 1$.
- Applying Chernoff bounds with $\delta = c - 1$ yields $\Pr[X_i > c] < \frac{e^{c-1}}{c^c}$
- Let $\gamma(n)$ be number x such that $x^x = n$, and choose $c = e\,\gamma(n)$.

$$\Pr[X_i > c] < \frac{e^{c-1}}{c^c} < \left(\frac{e}{c}\right)^c = \left(\frac{1}{\gamma(n)}\right)^{e\gamma(n)} < \left(\frac{1}{\gamma(n)}\right)^{2\gamma(n)} = \frac{1}{n^2}$$

- Union bound $\Rightarrow$ with probability $\geq 1 - 1/n$ no processor receives more than $e\,\gamma(n) = \Theta(\log n / \log\log n)$ jobs.

Load Balancing

Many jobs. Suppose the number of jobs m = 16n ln n. Then on average, each processor handles μ = 16 ln n jobs. The probability that any processor exceeds 2 μ jobs is at most 1/n.

Analysis.

- Let X_i, Y_{ij} be as before.
- Applying Chernoff bounds with $\delta = 1$ yields

$$\Pr[X_i > 2\mu] < \left(\frac{e}{4}\right)^{16n\ln n} < \left(\frac{1}{e}\right)^{\ln n} = \frac{1}{n^2} \qquad \Pr[X_i < \tfrac{1}{2}\mu] < e^{-\frac{1}{2}\left(\frac{1}{2}\right)^2(16n\ln n)} = \frac{1}{n^2}$$

- Applying the union bound, we conclude that all n machines will have load between half and twice the average.