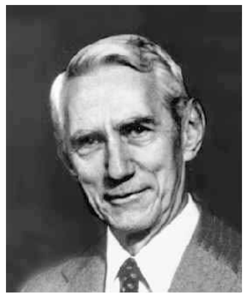
Lecture 10: Combinational Circuits

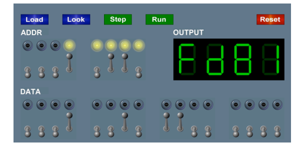
George Boole (1815 - 1864)



Claude Shannon (1916 - 2001)

Computer Architecture

Previous two lectures.
- TOY machine.



Next two lectures.
- Digital circuits.

Culminating lecture.
- Putting it all together and building a TOY machine.

Digital Circuits

What is a digital system?
- Digital: signals are 0 or 1.
- Analog: signals vary continuously.

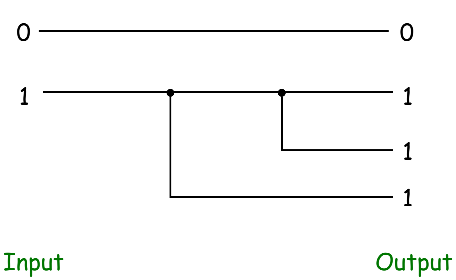
Why digital systems?
- Accuracy and reliability.
- Staggeringly fast and cheap.

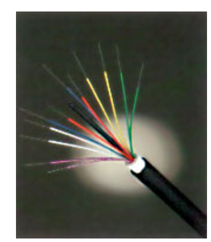
Basic abstractions.
- On, off.
- Switch that can turn something on or off.

Digital circuits and you.
- ➡ Computer microprocessors.
- Antilock brakes.
- Cell phones.

Wires

Wires.
- Propagate logical values from place to place.
- Signals "flow" from left to right.
  - A drawing convention, sometimes violated
  - Actually: flow from producer to consumer(s) of signal

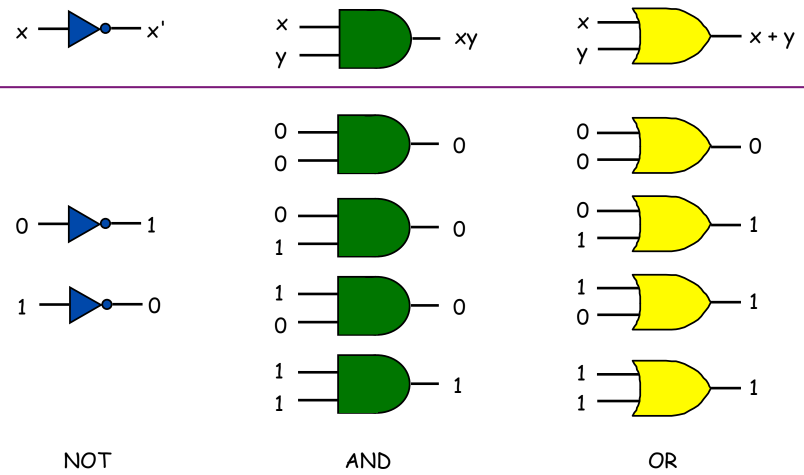




Logic Gates

Logical gates.

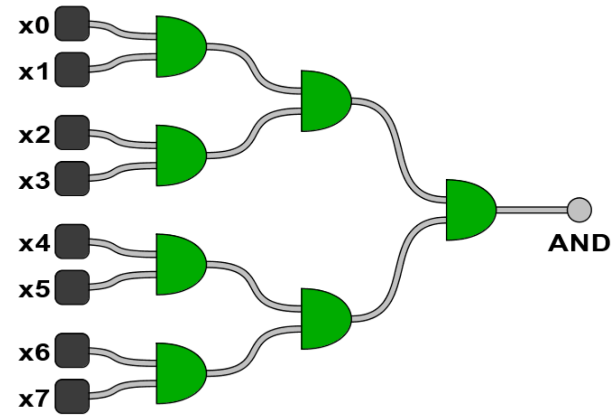
- Fundamental building blocks.



Multiway AND Gates

$AND(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$.

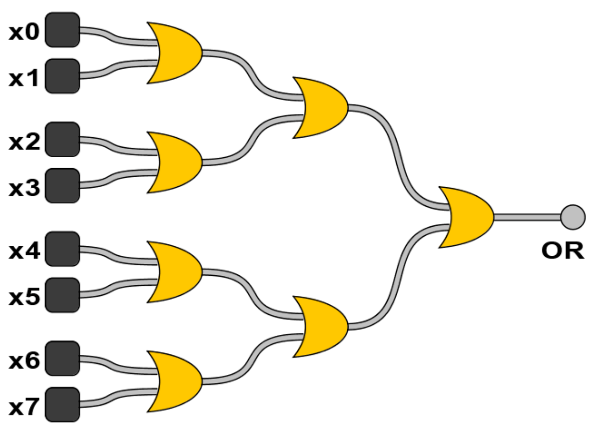
- 1 if all inputs are 1.
- 0 otherwise.



Multiway OR Gates

$OR(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$.

- 1 if at least one input is 1.
- 0 otherwise.



Boolean Algebra

History.

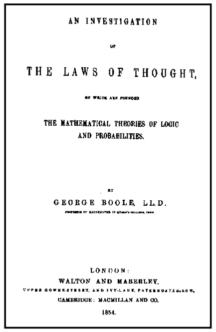
- Developed by Boole to solve mathematical logic problems (1847).
- Shannon first applied to digital circuits (1937).

Basics.

- Boolean variable: value is 0 or 1.
- Boolean function: function whose inputs and outputs are 0, 1.

Relationship to circuits.

- Boolean variables: signals.
- Boolean functions: circuits.


AN INVESTIGATION

THE LAWS OF THOUGHT,

THE MATHEMATICAL THEORIES OF LOGIC AND PROBABILITIES.

BY

GEORGE BOOLE, LL.D.

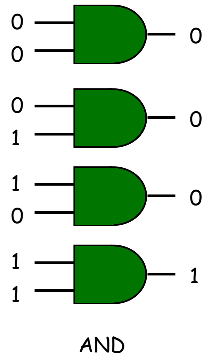
LONDON:
WALTON AND MABERLEY,
CAMBRIDGE: MACMILLAN AND CO.
1854.

Truth Table

Truth table.

- Systematic method to describe Boolean function.
- One row for each possible input combination.
- N inputs $\Rightarrow$ 2^N rows.

AND Truth Table		
x	y	AND(x, y)
0	0	0
0	1	0
1	0	0
1	1	1



Truth Table for Functions of 2 Variables

Truth table.

- 16 Boolean functions of 2 variables.
  - every 4-bit value represents one

Truth Table for All Boolean Functions of 2 Variables									
x	y	ZERO	AND		x		y	XOR	OR
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

Truth Table for All Boolean Functions of 2 Variables									
x	y	NOR	EQ	y'		x'		NAND	ONE
0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

Truth Table for Functions of 3 Variables

Truth table.

- 16 Boolean functions of 2 variables.
  - every 4-bit value represents one
- 256 Boolean functions of 3 variables.
  - every 8-bit value represents one
- 2^(2^N) Boolean functions of N variables!

Some Functions of 3 Variables						
x	y	z	AND	OR	MAJ	ODD
0	0	0	0	0	0	0
0	0	1	0	1	0	1
0	1	0	0	1	0	1
0	1	1	0	1	1	0
1	0	0	0	1	0	1
1	0	1	0	1	1	0
1	1	0	0	1	1	0
1	1	1	1	1	1	1

Universality of AND, OR, NOT

Any Boolean function can be expressed using AND, OR, NOT.

- "Universal."
- XOR(x,y) = xy' + x'y

Expressing XOR Using AND, OR, NOT							
x	y	x'	y'	x'y	xy'	x'y + xy'	XOR
0	0	1	1	0	0	0	0
0	1	1	0	1	0	1	1
1	0	0	1	0	1	1	1
1	1	0	0	0	0	0	0

Notation	Meaning
x'	NOT x
x y	x AND y
x + y	x OR y

Exercise: {AND, NOT}, {OR, NOT}, {NAND}, {AND, XOR} are universal.

Sum-of-Products

Any Boolean function can be expressed using AND, OR, NOT.

- Sum-of-products is systematic procedure.
  - form AND term for each 1 in truth table of Boolean function
  - OR terms together

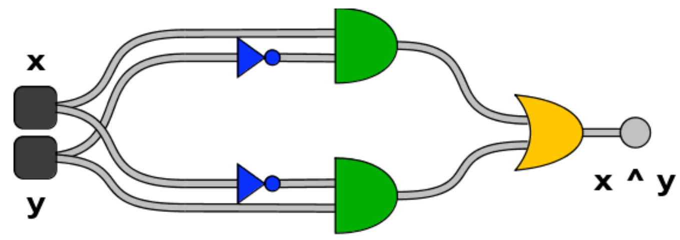
Expressing MAJ Using Sum-of-Products

x	y	z	MAJ	x'yz	xy'z	xyz'	xyz	x'yz + xy'z + xyz' + xyz
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	1
1	0	0	0	0	0	0	0	0
1	0	1	1	0	1	0	0	1
1	1	0	1	0	0	1	0	1
1	1	1	1	0	0	0	1	1

Translate Boolean Formula to Boolean Circuit

Use sum-of-products form.

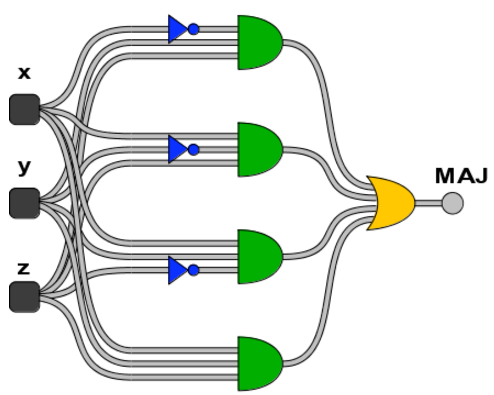
- XOR(x, y) = xy' + x'y.



Translate Boolean Formula to Boolean Circuit

Use sum-of-products form.

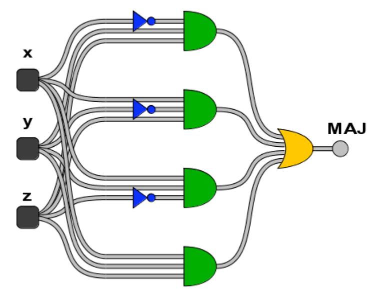
- MAJ(x, y, z) = x'yz + xy'z + xyz' + xyz.



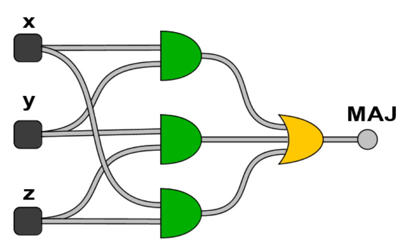
Simplification Using Boolean Algebra

Many possible circuits for each Boolean function.

- Sum-of-products not necessarily optimal in:
  - number of gates (space)
  - depth of circuit (time)
- MAJ(x, y, z) = x'yz + xy'z + xyz' + xyz = xy + yz + xz.



size = 8, depth = 3



size = 4, depth = 2

Expressing a Boolean Function Using AND, OR, NOT

Ingredients.
- AND gates.
- OR gates.
- NOT gates.
- Wire.

Instructions.
- Step 1: represent input and output signals with Boolean variables.
- Step 2: construct truth table to carry out computation.
- Step 3: derive (simplified) Boolean expression using sum-of products.
- Step 4: transform Boolean expression into circuit.

ODD Parity Circuit

ODD(x, y, z).
- 1 if odd number of inputs are 1.
- 0 otherwise.

Expressing ODD Using Sum-of-Products

x	y	z	ODD	x'y'z	x'yz'	xy'z'	xyz	x'y'z + x'yz' + xy'z' + xyz
0	0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0	1
0	1	0	1	0	1	0	0	1
0	1	1	0	0	0	0	0	0
1	0	0	1	0	0	1	0	1
1	0	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	1	1	1	0	0	0	1	1

ODD Parity Circuit

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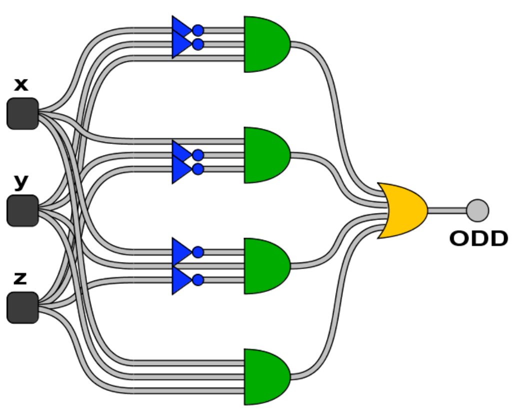


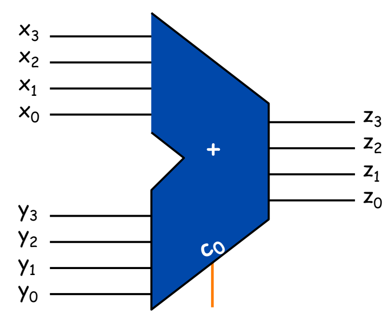
Let's Make an Adder Circuit

Goal: x + y = z for 4-bit integers.
- We build 4-bit adder: 9 inputs, 4 outputs.
- Same idea scales to 128-bit adder.
- Key computer component.

	1	1	1	0
	2	4	8	7
+	3	5	7	9
	6	0	6	6

Step 1.
- Represent input and output in binary.

	1	1	0	0
	0	0	1	0
+	0	1	1	1
	1	0	0	1



	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Let's Make an Adder Circuit

Goal: $x + y = z$ for 4-bit integers.

				c_0
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Step 2. (first attempt)

- Build truth table.
- Why is this a bad idea?
  - 128-bit adder: 2^{256+1} rows > # electrons in universe!

4-Bit Adder Truth Table

c_0	x_3	x_2	x_1	x_0	y_3	y_2	y_1	y_0	z_3	z_2	z_1	z_0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	1	1	0	0	1	1
0	0	0	0	0	0	1	0	0	0	1	0	0
0	0	0	0	0	0	1	0	1	0	1	0	1
.	.	.	.	.	.	.	.	.	.	.	.	.
1	1	1	1	1	1	1	1	1	1	1	1	1

2^{8+1} = 512 rows!

Let's Make an Adder Circuit

Goal: $x + y = z$ for 4-bit integers.

	c_3	c_2	c_1	$c_0 = 0$
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Step 2. (do one bit at a time)

- Build truth table for carry bit.
- Build truth table for summand bit.

Carry Bit

x_i	y_i	c_i	c_{i+1}
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Summand Bit

x_i	y_i	c_i	z_i
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Let's Make an Adder Circuit

Goal: $x + y = z$ for 4-bit integers.

	c_3	c_2	c_1	$c_0 = 0$
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Step 3.

- Derive (simplified) Boolean expression.

Carry Bit

x_i	y_i	c_i	c_{i+1}	MAJ
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

Summand Bit

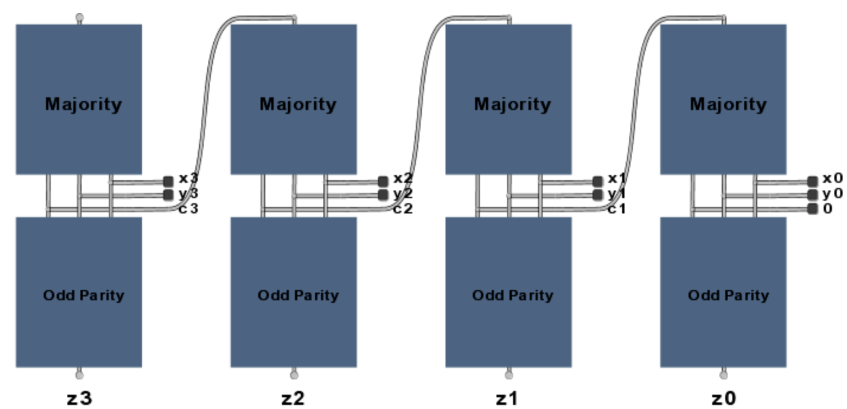
x_i	y_i	c_i	z_i	ODD
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	1	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

Let's Make an Adder Circuit

Goal: $x + y = z$ for 4-bit integers.

Step 4.

- Transform Boolean expression into circuit.
- Chain together 1-bit adders.



Let's Make an Adder Circuit

Goal: x + y = z for 4-bit integers.

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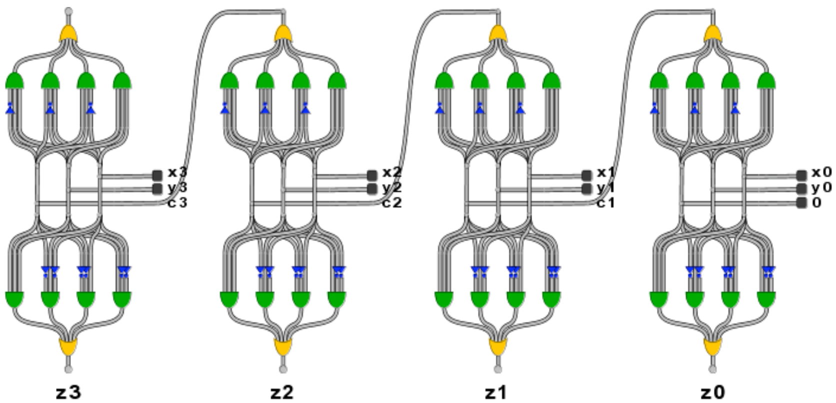
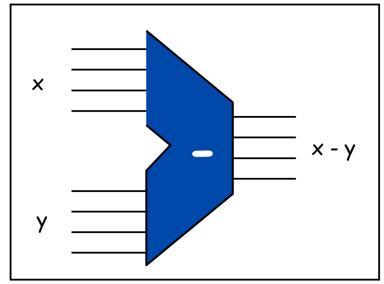


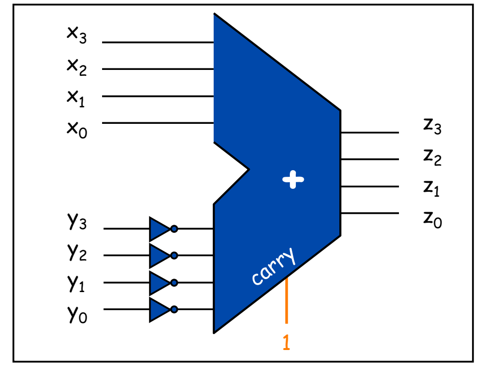
Subtractor

Subtractor circuit: z = x - y.

- One approach: design like adder circuit.
- Better idea: reuse adder circuit.
  - 2's complement: to negate an integer, flip bits, then add 1



4-Bit Subtractor Interface



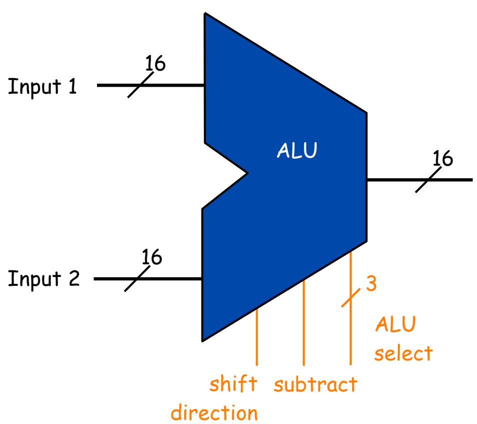
4-Bit Subtractor Implementation

Arithmetic Logic Unit: Interface

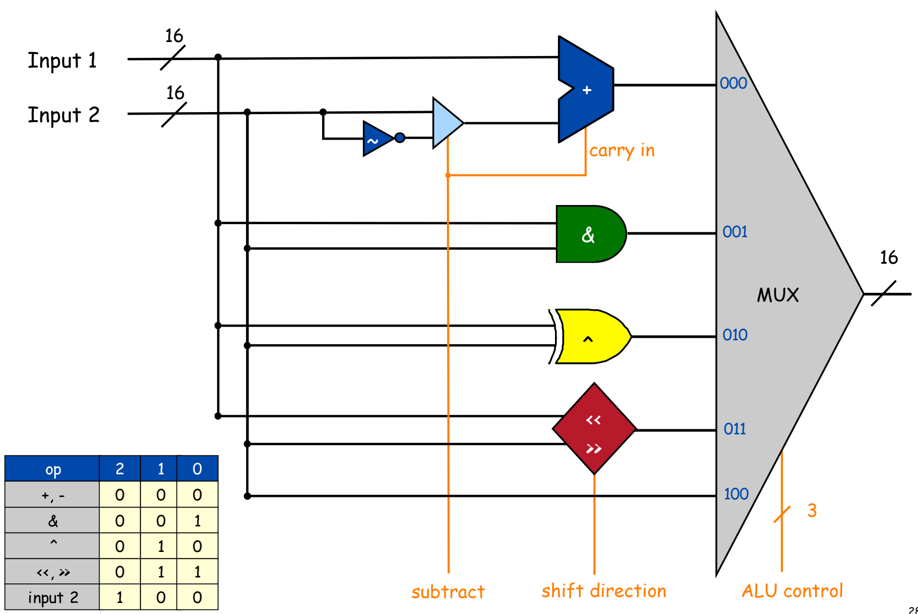
ALU Interface.

- Add, subtract, bitwise and, bitwise xor, shift left, shift right, copy.
- Associate 3-bit integer with 5 primary ALU operations.
  - ALU performs operations in parallel
  - control wires select which result ALU outputs

op	2	1	0
+, -	0	0	0
&	0	0	1
^	0	1	0
<<, >>	0	1	1
input 2	1	0	0



Arithmetic Logic Unit: Implementation



op	2	1	0
+, -	0	0	0
&	0	0	1
^	0	1	0
<<, >>	0	1	1
input 2	1	0	0

Summary

Lessons for software design apply to hardware design!

- Interface describes behavior of circuit.
- Implementation gives details of how to build it.

Layers of abstraction apply with a vengeance!

- On/off.
- Controlled switch (transistor).
- Gates (AND, OR, NOT).
- Boolean circuit (MAJ, ODD).
- Adder.
- . . .
- Arithmetic logic unit.
- . . .
- TOY machine.