Collaboration Policy: You must do problems 1 to 5 on your own without referring to texts/papers. Collaboration on 6 is allowed.

1. In this exercise, the goal is to prove the max flow mincut theorem using LP duality. Write down an LP formulation for the s-t maxflow in a given graph with capacities $c_e$ on edges. Write the dual of this LP and show that it is equal to the mincut between $s$ and $t$ in the graph.

2. Give a bound on the number of cuts of size at most $\alpha$ times the mincut in an undirected graph. (Your score will depend on how tight your bound is). Does your bound work for directed graphs?

3. Consider the bottleneck flow algorithm to compute a maxflow from $s$ to $t$ in a unit-capacity graph. Show that the algorithm spends $O(mk)$ time to increase the distance from $s$ to $t$ in the residual graph to $k$. Also show that when the distance between $s$ and $t$ is at least $k$, the mincut from $s$ to $t$ is $O(\min(n^2/k^2, m/k))$.

4. We discussed the mincost flow algorithm that finds a cycle $W$ such that the cost reduction obtained by augmenting along $W$ is at least $(OPT - C)/(m + n)$ where $OPT$ is the cost of an optimum solution and $C$ is the cost of the current solution. We noticed that finding such a cycle is NP-hard. We sketched an argument to find a set of cycles to obtain the same cost reduction using cycle covers. A cycle cover of a directed graph is a set of cycles such that every vertex is in exactly one of the cycles. A minimum cost cycle cover can be found in polynomial time. Show that you can use this algorithm to find a set of augmentations that reduce the cost of the current solution by at least $(OPT - C)/(m + n)$.

5. Suppose we have a partial order on $n$ elements. We can represent the partial order as a directed acyclic graph. A chain in a partial order is simply a subset of elements that are totally ordered (they induce a path in the graph). An anti-chain is a set of elements that are incomparable (they form an independent set in the graph). Dilworth’s theorem for partial orders is the following: in any partial order the minimum number $m$ of disjoint chains which together contain all the elements is equal to the maximum number $M$ of elements contained in an antichain. Prove Dilworth’s theorem using the maxflow-mincut theorem.
6. **Bonus question:** Some students claimed they could achieve a runtime of $O(m \cdot \text{polylog}(n))$ in the randomized min cut algorithm by a clever implementation of the contraction step. If you come up with such an implementation, you will receive an automatic A+. 