COS 521: Advanced Algorithm Design
Final
Due: Mon, May 17 (5 pm)

Instructions: There are a total of 6 questions on the final. No collaboration is allowed. You may refer to any reference material; however you should not explictly search for solutions to the questions on the exam. If you do refer to any sources, indicate this on your homework.

1. Given a directed graph $G = (V, E)$, let $f(G)$ be the minimum number of edges of G that need to be deleted so that the remaining graph does not contain any directed cycle.

   (i) Let $\mathcal{D}(G)$ be a collection of disjoint directed cycles in G. Show that $|\mathcal{D}(G)| \leq f(G)$. (Note: $|\mathcal{D}(G)|$ is the number of cycles in $\mathcal{D}(G)$).

   (ii) Consider a fractional version of the previous bound. Let $\mathcal{D}'(G)$ be a collection of disjoint directed cycles in G, where each cycle $C \in \mathcal{D}'(G)$ is associated with a non-negative weight w_C. Further for every edge e, the sum of weights of all cycles containing e is at most 1. Let $W = \sum w_C$ be the sum of weights of cycles in $\mathcal{D}'(G)$. Show that $W \leq f(G)$.

   (iii) Consider the following LP:

$$\begin{array}{rrcl} & \min \sum_{uv \in E} x_{uv} & & \\ \forall\, u, v & x_{uv} + x_{vu} & = & 1 \\ \forall\, u, v, w & x_{uv} + x_{vw} + x_{wu} & \geq & 1 \\ \forall\, u, v & x_{uv} & \geq & 0 \end{array}$$

   Let OPT_{LP} be the optimal value of the LP above. Show that $OPT_{LP} \leq f(G)$.

   (iv) Show that the best possible bound obtainable in part (ii) is equal to OPT_{LP}.

2. Let $G = (V, E)$ be an undirected graph and let c be the value of the minimum cut in G. Consider the graph G_p obtained by sampling every edge of G with probability p, i.e. for each edge e of G, e is placed in G_p independently with probability p. Show that for $p = \Omega((\ln n)/(\epsilon^2 c))$ every cut in G_p has value between $1 - \epsilon$ and $1 + \epsilon$ times its expected value with very high probability. (*Hint:* In an undirected graph the number of α-minimum cuts (i.e. cuts with value at most α times the minimum cut value) is at most $n^{2\alpha}$. You may use this without proof.)

3. Let $G = (V, E)$ be a simple undirected graphs and let k be a positive integer. Call a vertex v in G a low-degree vertex if $d(v) \leq n/k$. Consider a pair of vertices (s, t) such that $\text{dist}(s, t) \geq k$. Show that every $s - t$ path has $\Omega(k)$ low degree vertices on it.

4. Consider a directed graph $G = (V, A)$ that has integer (can be negative) costs $c(a)$ on each arc a and a positive integer weight $p(v)$ on each vertex v. The mean cost of a cycle C in G is $\sum_{a \in C} c(a) / \sum_{v \in C} p(v)$. Show how to compute the minimum mean cost cycle in polynomial time.

5. Consider a hypergraph $G = (V, E)$. Each edge $e \in E$ is a subset of V. A sub-hypergraph $H = (V_H, E_H)$ has edge-density λ if $|E_H|/|V_H| \geq \lambda$. Show that you can compute the maximum density sub-hypergraph of G in polynomial time using a reduction to maxflow. Note that the maximum density sub-hypergraph can be G itself.

6. Given a graph $G = (V, E)$, and a set S of terminals ($S \subseteq V$), we would like to partition the set of vertices into disjoint sets such that all terminals are separated (i.e. no two terminals are in the same set of the partition). The objective is to minimize the number of edges cut. Consider the following algorithm for this problem: For each terminal $t \in S$, let C_t be the minimum cut separating t from $S - \{t\}$. To obtain the final solution, we take the union of the $k - 1$ smallest such cuts. Give an upper bound on the approximation ratio of this algorithm.