Dynamic Trees

- Motivation (online MSTs)
- Problem Definition
- A Data Structure for Dynamic Paths
- A Data Structure for Dynamic Trees
- Extensions

Online Minimum Spanning Trees

- The online minimum spanning trees problem:
  - Input: a sequence of edges (with costs), one at a time.
  - Goal: keep the minimum spanning forest of the graph.
- An algorithm:
  - For each new edge $(u, v)$:
    - If $u$ and $v$ belong to different components, insert the edge.
    - If $u$ and $v$ are in the same component:
      - Insert $(u, v)$ into the union, and
      - Remove the most expensive edge in the cycle created.
Renato Werneck

Dynamic Trees
Online Minimum Spanning Trees

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Dynamic Trees
Online Minimum Spanning Trees

<table>
<thead>
<tr>
<th>edge</th>
<th>cost</th>
</tr>
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<tbody>
<tr>
<td>(a,b)</td>
<td>6</td>
</tr>
<tr>
<td>(a,c)</td>
<td>3</td>
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<tr>
<td>(a,d)</td>
<td>4</td>
</tr>
<tr>
<td>(b,d)</td>
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<tr>
<td>(b,e)</td>
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<tr>
<td>(c,d)</td>
<td>1</td>
</tr>
<tr>
<td>(d,e)</td>
<td>2</td>
</tr>
<tr>
<td>(e,f)</td>
<td>2</td>
</tr>
</tbody>
</table>

• How fast is the algorithm?
  - How fast can we find the most expensive edge in a cycle?
    - \(O(\log n)\), with the right data structure.
  - Total running time: \(O(m \log n)\) \((m \text{ edges}, n \text{ vertices})\)

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Dynamic Trees - Problem Definition

- Goal: maintain a forest of rooted trees with costs on vertices.
  - Each tree has a root, every edge directed towards the root.
- Operations allowed:
  - \(\text{link}(u,v)\): creates an edge between \(u\) (a root) and \(v\).
  - \(\text{cut}(v)\): deletes edge \((u, p(v))\) (where \(p(v)\) is its parent).
  - \(\text{findcost}(v)\): returns the cost of vertex \(v\).
  - \(\text{findroot}(v)\): returns the root of the tree containing \(v\).
  - \(\text{findmin}(v)\): returns the vertex \(w\) of minimum cost in the path from \(u\) to the root.
- A possible extension:
  - \(\text{event}(u)\): makes \(u\) the root of its tree

Dynamic Trees

- An example (two trees):

![Dynamic Trees Diagram]
Dynamic Trees

- **link:**
  - $E_{11}$
  - $E_{12}$

- **cut:**
  - $E_{21}$
  - $E_{22}$

Dynamic Trees

- **findmin:** $s \rightarrow b$
- **findroot:** $s \rightarrow a$
- **findcost:** $s \rightarrow 2$

Applications

- Used as a building block of several graph algorithms:
  - online minimum spanning trees
  - dynamic graphs
  - directed minimum spanning trees
  - network flows (e.g., maximum flow)
  - ...

Dynamic Trees and Online MSTs

- **How can dynamic trees help us in the online MST problem?**
  - We must answer the following (equivalent) questions:
    - Should we insert $(c,g)$, with cost 4, into the following tree?
    - Is $(c,g)$ cheaper than some other edge in the cycle it creates?
    - What is the most expensive edge in the path between $c$ and $g$?

Dynamic Trees and Online MST

- **How can dynamic trees help us in the online MST problem?**
  - We must answer the following (equivalent) questions:
    - Should we insert $(c,g)$, with cost 4, into the following tree?
    - Is $(c,g)$ cheaper than some other edge in the cycle it creates?
    - What is the most expensive edge in the path between $c$ and $g$?
    - Imagine the tree is rooted at $g$ now, what is the most expensive edge in the path from $c$ to the root?
Obvious Implementation of Dynamic Trees

- Each node represents a vertex.
- Each node \( x \) points to its parent \( p(x) \):
  - cut, link, findroot: constant time.
  - findroot, findmin: time proportional to path length.
- Acceptable if paths are small, but \( O(n) \) in the worst case.
- We can get \( O(\log n) \) for all operations.

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Dynamic Paths

- We start with a simpler problem:
  - Maintain set of paths subject to the following operation:
    - \( \text{split} \): removes an edge, cutting a path in two;
    - \( \text{concatenate} \): links endpoints of two paths, creating a new path.
  - Operations allowed:
    - \( \text{findmin}(v) \): returns the root of vertex \( v \);
    - \( \text{findmin}(e) \): returns minimum-cost vertex in the path containing \( e \).

Simple Paths as Lists

- Natural representation: doubly-linked list:
  - Path characterized by two endpoints:
    - \( \text{findroot} \): constant time.
    - \( \text{concatenate} \): constant time.
    - \( \text{split} \): constant time.
    - \( \text{findmin} \): linear time (not good).
  - Can we do it \( O(\log n) \) time?

Simple Paths as Binary Trees

- Alternative representation: balanced binary tree.
  - Leaves: vertices in symmetric order.
  - Internal nodes: subpaths between extreme descendants.

Simple Paths as Binary Trees

- Compact alternative:
  - Each internal node represents both a vertex and a subpath:
    - subpath from leftmost to rightmost descendant.
Simple Paths: Maintaining Costs

- We store cost(x) directly on each vertex.
- Problem: findmin still takes linear time (must visit all vertices).

Simple Paths: Finding Minima

- Also store mincost(x), minimum cost in subpath with root x.
- findmin(x) now runs in O(log n) time.

Simple Paths: Data Fields

- Final version:
  - Stores mincost(x) and cost(x) for every vertex x.

Simple Paths: Structural Changes

- Concatenating and splitting paths:
  - Join or split the corresponding binary trees;
  - Time proportional to tree height.
  - For balanced trees (AVL, red-black, etc.), this is O(log n):
    - Rotations must be supported in constant time.
    - We must be able to update mincost, but that's easy:

Splaying

- Simpler alternative to balanced binary trees: splaying.
  - Does not guarantee that trees are balanced in the worst case.
  - Guarantees O(log n) access in the amortized sense.
  - Makes the data structure much simpler to implement.
  - Basic characteristics:
    - Does not require any balancing information;
    - On an access to v:
      - Moves v to the root;
      - Roughly halves the depth of other nodes in the access path.
    - Primitive operation: rotation.
  - All operations (insert, delete, join, split) use splaying.

Splaying

- Three restructuring operations:
An Example of Splaying

Dynamic Trees

An Example of Splaying

Dynamic Trees

An Example of Splaying

Dynamic Trees

An Example of Splaying

Dynamic Trees

Dynamic Trees
Amortized Analysis

- Bounds the running time of a sequence of operations.
- Potential function $\Phi$ maps configurations to real numbers.
- Amortized time to execute each operation:
  - $c_i = t_i + \Phi_i - \Phi_{i-1}$
    - $t_i$: amortized time to execute $i$:th operation;
    - $\Phi_i$: potential after the $i$:th operation.
- Total time for $m$ operations:
  \[ \sum_{i=1}^{m} t_i = \sum_{i=1}^{m} (c_i + \Phi_i - \Phi_{i-1}) = \Phi_m - \Phi_1 + \sum_{i=1}^{m} c_i \]

Amortized Analysis of Splaying

- Definitions:
  - $s(x)$: size of node $x$ (number of descendants, including $x$);
    - At most $n$, by definition.
  - $r(x)$: rank of node $x$, defined as $\log s(x)$;
    - At most $\log n$, by definition.
  - $\Phi$: potential of the data structure (twice the sum of all ranks).
    - At most $n \log n$, by definition.
- Access Lemma [ST83]: The amortized time to splay a tree with root $t$ at a node $x$ is at most
  \[ 6(r(t) - r(x)) + 1 = O(\log s(x)/s(x)) \]
Proof of Access Lemma
- Access Lemma [SF87]: The amortized time to splay a tree with root at a node *x* is at most
  \[ \Theta(r(x) - r(x)) + 1 = O(\log(n)/\Theta(x)) \]
- Proof idea:
  - \( r(x) \) = rank of *x* after the i-th splay step;
  - \( c_i \) = amortized cost of the i-th splay step;
  - \( n_i \leq \Theta(r(x)) - \Theta(x) + 1 \) (for the zig-zag step, if any)
  - \( n_i \leq \Theta(r(x)) - \Theta(x) \) (for any zig-zag and zig-zag steps)
  - Total amortized time for all *k* steps:
    \[ \sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} \Theta(r(x)) - \Theta(x) \leq k \leq 6 \Theta(x) - \Theta(x) + 1 \]
    \[ = 6n_i - 6n_i + 1 \]

Proof of Access Lemma: Splaying Step
- Zig-zag:
  
  Claim: \( a \leq 4 (\Theta(x)) - \Theta(x) \)
  \[ t = \Theta(x) - \Theta(x) \]
  \[ 2 + (2\Theta(x) + 2\Theta(y) - 2\Theta(x)) \leq 2 + (2\Theta(x) + 2\Theta(y) - 2\Theta(x)) \leq 4 (\Theta(x)) - \Theta(x) \]
  \[ = 2 + (2\Theta(y) - 2\Theta(x)) \leq 4 (\Theta(x)) - \Theta(x), \] since \( \Theta(y) \geq \Theta(x) \)
  \[ (\Theta(y)) - \Theta(x)) + \Theta(y)) = \leq 1, \] rearranging
  \[ \log(\Theta(y)) - \Theta(x)) + \Theta(y)) = \leq 1 \]
  - TRUE because \( \Theta(y)) - \Theta(x)) \) both ratios are smaller than 1, at least one
  is at most \( 1/2 \) (and its log is at most \( -1 \)).

Proof of Access Lemma: Splaying Step
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  is at most \( 1/2 \) (and its log is at most \( -1 \)).

Splaying
- Summing up:
  - Notation: \( a = 1 \)
  - Zig: \( a \leq 6 (\Theta(x)) - \Theta(x) + 1 \)
  - Zig-zig: \( a \leq 6 (\Theta(x)) - \Theta(x) \)
  - Zig-zag: \( a \leq 4 (\Theta(x)) - \Theta(x) \)
  - Total amortized time at most \( (\Theta(x)) - \Theta(x) + 1 = O(\log(n)) \)
  - Since accesses bring the relevant element to the root, other operations (insert, delete, join, split) become trivial.

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Dynamic Trees

- We know how to deal with isolated paths.
- How to deal with paths within a tree?

Dynamic Trees

- Main idea: partition the vertices in a tree into disjoint solid paths connected by dashed edges.

Dynamic Trees

- A vertex \( v \) is exposed if:
  - There is a solid path from \( v \) to the root;
  - No solid edge enters \( v \).

Dynamic Trees

- A vertex \( v \) is exposed if:
  - There is a solid path from \( v \) to the root;
  - No solid edge enters \( v \);
  - It is unique.

Dynamic Trees

- Solid paths:
  - Represented as binary trees (as seen before).
  - Parent pointer of root is the outgoing dashed edge of the path.
    - Dashed pointers go up, so the solid path above does not “know” it has dashed children.
  - Solid binary trees linked by dashed edges: virtual tree.
  - “Isolated path” operations handle the exposed path.
    - That’s the solid path entering the root.
  - If a different path is needed:
    - expose(\( v \)): make entire path from \( v \) to the root solid.

Virtual Tree: An Example
Dynamic Trees

- Example: expose(y)

  - Take all edges in the path to the root, ...

Dynamic Trees

- Example: expose(y)

  - ... make them solid, ...

Dynamic Trees

- Example: expose(y)

  - ... make sure there is no other solid edge incident into the path.
  - Uses splice operation.

Exposing a Vertex

- expose(y): makes the path from y to the root solid.
- Implemented in three steps:
  1. Splay within each solid tree in the path from x to root.
  2. Splice each dashed edge from x to the root.
     - splice replaces left solid child with dashed child.
  3. Splay on x, which will become the root.

Exposing a Vertex: An Example

- expose(y): (1) splay within each solid tree;
- Does not change the partition into solid paths.
Exposing a Vertex: An Example

- expose(y): (c) splice on all vertices from y to the root.
  - Original exposed path: (d l i f c b a)
  - New exposed path: (y u t s mj g d c b a)

Dynamic Trees

Exposing a Vertex: An Example

- expose(y): (g) splay on y.
  - Does not change the exposed path.

Dynamic Trees

Dynamic Trees: Splice

- Additional restructuring primitive: splice.
  - Dashed child becomes solid, replaces left child.

Dynamic Trees

Exposing a Vertex: Running Time

- Running time of expose(x):
  - Proportional to initial depth of x;
    - x is rotated all the way to the root;
    - we just need to count the number of rotations.
  - Will use the Access Lemma:
    - $a(x), r(x)$ and potential are defined as before;
    - In particular, $a(x)$ is the size of the whole subtree rooted at x.
      - Includes both solid and dashed edges.

Dynamic Trees

Exposing a Vertex: Running Time (Proof)

- $k$: number of dashed edges from x to the root t.
- Amortized costs of each pass:
  1. Splay within each solid tree:
     - $x$: vertex splayed on the j-th solid tree.
     - amortized cost of j-th splay: $6 \cdot (j, x) = \delta(x) + 1$ (Access Lemma)
     - $r(x) \leq r(x_j)$, so the sum over a subtree does not pass.
     - amortized cost of first pass $6 \cdot (x \rightarrow \delta(x_j)) = k \leq 6 \log n = k$.
  2. Splay dashed edges:
     - no rotations, no changes in potential: amortized cost is zero.
  3. Splay on x:
     - amortized cost is at most $6 \log n = 1$.
     - splay at root, so exactly 3 rotations happen.
     - each rotation costs one credit, but charges two;
     - the payback is extra rotations in the first pass.
- Amortized number of rotations $s = O(\log n)$.

Dynamic Trees

Implementing Dynamic Tree Operations

- findcost(u):
  - expose u, return cost(u).
- findroot(u):
  - expose u;
    - find w, the rightmost vertex in the solid subtree containing v;
    - splay at w and return w.
- findmin(u):
  - expose u;
    - use mincost to walk down from x to w, the last minimum-cost node in the solid subtree containing v;
    - splay at w and return w.

Dynamic Trees
Implementing Dynamic Tree Operations

- link(u, w):
  - expose u and w (they are in different trees);
  - set p(u) = w (that is, make v a middlechild of w).
- cut(u):
  - expose v;
  - make p(right(v)) = null and right(v) = null;
  - set mincost(v) = min{cost(v), mincost(left(v))}.

Alternative Implementations

- Total time per operation depends on path representation:
  - Splay trees: O(log n) amortized [ Sleator and Tarjan, 83].
  - Balanced search trees: O(log n) amortized [ST83].
  - Locally biased search trees: O(log n) amortized [ST83].
  - Globally biased search trees: O(log n) worst-case [ST83].
- Biased search trees:
  - Support leaves with different "weights".
  - Some solid leaves are "heavier" because they also represent dashed subtrees.
  - Much more complicated than splay trees.

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Extension: Adding Costs

- addcost(v, x) adds x to the cost of all vertices in the path from v to the root.

Adding Costs to Dynamic Paths

- Corresponding operation on dynamic paths:
  - addcost(u, x): adds x to the cost of vertices in path containing v;
  - current representation takes linear time.

Adding Costs to Dynamic Paths

- Better approach is to store Δcost(x) instead (difference form):
  - Root: Δcost(x) = cost(x)
  - Other nodes: Δcost(x) = cost(x) - cost(p(x))
Adding Costs to Dynamic Paths

- Costs:
  - \( \text{addcost: constant time (just add to root)} \)
  - Finding cost(x) is slightly harder: \( O(\text{depth}(x)) \).

Adding Costs to Dynamic Paths

- Still have to implement findmin:
  - Cannot store mincost(x) explicitly (addcost would be linear).

Adding Costs to Dynamic Paths

- Store \( \text{mincost}(x) = \text{cost}(x) - \text{mincost}(x) \) instead.
  - findmin still runs in \( O(\log n) \) time, addcost now constant.

Adding Costs to Dynamic Paths: Updating Fields

- Updating fields during rotations:
  - \( \Delta \text{cost}(v) = \Delta \text{cost}(u) + \Delta \text{cost}(w) \)
  - \( \Delta \text{cost}(u) = -\Delta \text{cost}(v) \)
  - \( \Delta \text{cost}(b) = \Delta \text{cost}(v) + \Delta \text{cost}(b) \)
  - \( \Delta \text{min}(w) = \max(\Delta \text{min}(b), \Delta \text{min}(c)) - \Delta \text{cost}(v) \)
  - \( \Delta \text{min}(c) = \max(\Delta \text{min}(b), \Delta \text{min}(c)) - \Delta \text{cost}(w) - \Delta \text{cost}(u) \)

Adding Costs: Updating Fields

- Updating fields during splices:
  - \( \Delta \text{cost}(u) = \Delta \text{cost}(v) - \Delta \text{cost}(x) \)
  - \( \Delta \text{cost}(u) = \Delta \text{cost}(w) + \Delta \text{cost}(x) \)
  - \( \Delta \text{min}(x) = \max(\Delta \text{min}(b), \Delta \text{min}(c)) - \Delta \text{cost}(v) + \Delta \text{cost}(w) \)
  - \( \Delta \text{min}(x) = \Delta \text{min}(b) - \Delta \text{cost}(w) \)
  - Recall that \( w \) is always the root of a solid tree.
Renato Werneck

### Adding Costs: Operations

- **findroot(v):**
  - expose v, return \( \Delta \text{cost}(v) \).
- **findroot(w):**
  - expose v;
  - find w, the rightmost vertex in the solid subtree containing v;
  - splay at w and return w.
- **findmin(v):**
  - expose v;
  - use \( \Delta \text{cost} \) and \( \Delta \text{min} \) to walk down from v to w, the last minimum-cost node in the solid subtree;
  - splay at w and return w.

### Adding Costs: Operations

- **addroot(v, x):**
  - expose v;
  - add x to \( \Delta \text{cost}(v) \), subtract x from \( \Delta \text{cost}(\text{left}(v)) \);
  - link(v, w):
    - expose v and w (they are in different trees);
    - set \( \text{p}(v) = w \) (that is, make v a middle child of w).
  - cut(v):
    - expose w;
    - add \( \Delta \text{cost}(v) \) to \( \Delta \text{cost}(\text{right}(v)) \);
    - make \( \text{p}(\text{right}(v)) = \text{null} \) and \( \text{right}(v) = \text{null} \).
    - set \( \Delta \text{min}(v) = \max \{0, \Delta \text{min}(\text{left}(v)) - \Delta \text{cost}(\text{left}(v))\} \)

### Another Extension: Change Root

- **event(q):** makes q the root of its tree
  - Make sure q is exposed, reverse solid path.

### Another Extension: Change Root

- **event(q):** makes q the root of its tree
  - In the virtual tree, reverse left-right pointers:
    - This can be done implicitly with a reverse bit.
    - Must be stored in difference form (meaning depends on parent).
  - **event(q):** makes q the root of its tree

### Other Extensions

- Associate values with edges:
  - just interpret \( \Delta \text{cost}(v) \) as \( \Delta \text{cost}(v, p(v)) \).
- Other path queries (such as length):
  - modify values stored in each node appropriately.
- Free (unrooted) trees: use event to change root.
- Subtree-related operations:
  - Can be implemented, but parent must have access to middle children in constant time.
  - Tree must have bounded degree.
  - Approach for arbitrary trees: “ternarize” them
    - [Goldberg, Grigoriadis and Tarjan, 1991]