Max Flow, Min Cut

- Minimum cut
- Maximum flow
- Max-flow min-cut theorem
- Ford-Fulkerson augmenting path algorithm
- Edmonds-Karp heuristics
- Bipartite matching

**Maximum Flow and Minimum Cut**

Max flow and min cut.
- Two very rich algorithmic problems.
- Cornerstone problems in combinatorial optimization.
- Beautiful mathematical duality.

Nontrivial applications / reductions.
- Network connectivity.
- Bipartite matching.
- Data mining.
- Open-pit mining.
- Airline scheduling.
- Image processing.
- Project selection.
- Baseball elimination.
- Network reliability.
- Security of statistical data.
- Distributed computing.
- Egalitarian stable matching.
- Distributed computing.
- Many many more . . .

**Network:** abstraction for material FLOWING through the edges.
- Directed graph.
- Capacities on edges.
- Source node s, sink node t.

**Min cut problem.** Delete "best" set of edges to disconnect t from s.

**Minimum Cut Problem**

A cut is a node partition \((S, T)\) such that \(s\) is in \(S\) and \(t\) is in \(T\).

- \(\text{capacity}(S, T) = \text{sum of weights of edges leaving } S.\)

**Minimum Cut Problem**

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**Maximum Flow Problem**

Network: abstraction for material FLOWING through the edges.

- Directed graph.
- Capacities on edges.
- Source node \(s\), sink node \(t\).

Max flow problem. **Assign flow to edges so as to:**

- Equalize inflow and outflow at every intermediate vertex.
- Maximize flow sent from \(s\) to \(t\).
A flow $f$ is an assignment of weights to edges so that:

- **Capacity:** $0 \leq f(e) \leq u(e)$.
- **Flow conservation:** flow leaving $v = \text{flow entering } v$ except at $s$ or $t$.

**Flows**

```
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   / \
  /   \
 /     \
4     10
/       \
5       \
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4         \
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```

Value = 4

**Max flow problem:** find flow that maximizes net flow into sink.

**Flows and Cuts**

**Observation 1.** Let $f$ be a flow, and let $(S, T)$ be any $s$-$t$ cut. Then, the net flow sent across the cut is equal to the amount reaching $t$.
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Flows and Cuts

Observation 2. Let $f$ be a flow, and let $(S, T)$ be any s-t cut. Then the value of the flow is at most the capacity of the cut.

**Cut capacity = 30 $\Rightarrow$ Flow value $\leq 30$**

Max Flow and Min Cut

Observation 3. Let $f$ be a flow, and let $(S, T)$ be an s-t cut whose capacity equals the value of $f$. Then $f$ is a max flow and $(S, T)$ is a min cut.

**Cut capacity = 28 $\Rightarrow$ Flow value $\leq 28$**

Flow value = 28
Max-Flow Min-Cut Theorem

Max-flow min-cut theorem. (Ford-Fulkerson, 1956): In any network, the value of max flow equals capacity of min cut.

- Proof IOU: we find flow and cut such that Observation 3 applies.

Min cut capacity = 28  ⇔  Max flow value = 28

Towards an Algorithm

Find s-t path where each arc has \( f(e) < u(e) \) and "augment" flow along it.

- Greedy algorithm: repeat until you get stuck.

Towards an Algorithm

Find s-t path where each arc has \( f(e) < u(e) \) and "augment" flow along it.

- Greedy algorithm: repeat until you get stuck.
- Fails: need to be able to "backtrack."

Bottleneck capacity of path = 10

Flow value = 0

Flow value = 10

Flow value = 14
**Residual Graph**

Original graph.
- Flow \( f(e) \).
- Edge \( e = v-w \)

Residual edge.
- Edge \( e = v-w \) or \( w-v \).
- "Undo" flow sent.

Residual graph.
- All the edges that have strictly positive residual capacity.

**Augmenting Paths**

Augmenting path = path in residual graph.
- Increase flow along forward edges.
- Decrease flow along backward edges.

**Ford-Fulkerson Augmenting Path Algorithm**


Observation 4. If augmenting path, then not yet a max flow.
Q. If no augmenting path, is it a max flow?

Questions.
- Does this lead to a maximum flow? yes
- How do we find an augmenting path? s-t path in residual graph
- How many augmenting paths does it take?
- How much effort do we spending finding a path?
Max-Flow Min-Cut Theorem

Augmenting path theorem. A flow $f$ is a max flow if and only if there are no augmenting paths.

**Max-flow min-cut theorem.** The value of the max flow is equal to the capacity of the min cut.

We prove both simultaneously by showing the following are equivalent:

(i) $f$ is a max flow.
(ii) There is no augmenting path relative to $f$.
(iii) There exists a cut whose capacity equals the value of $f$.

(i) $\implies$ (ii) equivalent to not (ii) $\implies$ not (i), which was Observation 4
(ii) $\implies$ (iii) next slide
(iii) $\implies$ (i) this was Observation 3

Proof of Max-Flow Min-Cut Theorem

(ii) $\implies$ (iii). If there is no augmenting path relative to $f$, then there exists a cut whose capacity equals the value of $f$.

Proof.
- Let $f$ be a flow with no augmenting paths.
- Let $S$ be set of vertices reachable from $s$ in residual graph.
  - $S$ contains $s$: since no augmenting paths, $S$ does not contain $t$
  - all edges $e$ leaving $S$ in original network have $f(e) = u(e)$
  - all edges $e$ entering $S$ in original network have $f(e) = 0$

$$|f| = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ in to } S} f(e) = \sum_{e \text{ out of } S} u(e) = \text{capacity}(S, T)$$

Max Flow Network Implementation

Edge in original graph may correspond to 1 or 2 residual edges.
- May need to traverse edge $e = v-w$ in forward or reverse direction.
- Flow $= f(e)$, capacity $= u(e)$.
- Insert two copies of each edge, one in adjacency list of $v$ and one in $w$.

```java
public class Edge {
    private int v, w; // from, to
    private int cap; // capacity from v to w
    private int flow; // flow from v to w

    public Edge(int v, int w, int cap) { ... }

    public int cap() { return cap; }
    public int flow() { return flow; }
    public boolean from(int v) { return this.v == v; }
    public int other(int v) { return from(v)? this.w : this.v; }
    public int capRto(int v) { return from(v)? flow : cap - flow; }
    public void addflowRto(int v, int d) { flow += from(v)?- d : d; }
}
```

Ford-Fulkerson Algorithm: Implementation

Ford-Fulkerson main loop.

```java
// while there exists an augmenting path, use it
while (augpath()) {

    // compute bottleneck capacity
    int bottle = INFINITY;
    for (int v = t; v != s; v = ST(v))
        bottle = Math.min(bottle, pred[v].capRto(v));

    // augment flow
    for (int v = t; v != s; v = ST(v))
        pred[v].addflowRto(v, bottle);

    // keep track of total flow sent from s to t
    value += bottle;
}
```
Ford-Fulkerson Algorithm: Analysis

Assumption: all capacities are integers between 1 and U.

Invariant: every flow value and every residual capacities remain an integer throughout the algorithm.

Theorem: the algorithm terminates in at most $|f*| \leq VU$ iterations.

Corollary: if $U = 1$, then algorithm runs in $\leq V$ iterations.

Integrality theorem: if all arc capacities are integers, then there exists a max flow $f$ for which every flow value is an integer.

Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

Original Network

Choosing Good Augmenting Paths

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Original Network
**Choosing Good Augmenting Paths**

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.
- Optimal choices for real world problems??

**Design goal is to choose augmenting paths so that:**

- Can find augmenting paths efficiently.
- Few iterations.

**Choose augmenting path with:**

- Edmonds-Karp (1972) (shortest path)
- Max bottleneck capacity. (fattest path)

**Shortest Augmenting Path**

Shortest augmenting path.

- Easy to implement with BFS.
- Finds augmenting path with fewest number of arcs.

```java
while (!q.isEmpty()) {
    int v = q.dequeue();
    IntIterator i = G.neighbors(v);
    while (i.hasNext()) {
        Edge e = i.next();
        int w = e.other(v);
        if (e.capRto(w) > 0) {
            // is v-w a residual edge?
            if (wt[w] > wt[v] + 1) {
                wt[w] = wt[v] + 1;
                pred[w] = e;  // keep track of shortest path
                q.enqueue(w);
            }
        }
    }
}
return (wt[t] < INFINITY);  // is there an augmenting path?
```
Shortest Augmenting Path Analysis

Length of shortest augmenting path increases monotonically.
- Strictly increases after at most $E$ augmentations.
- At most $E \cdot V$ total augmenting paths.
- $O(E^2 \cdot V)$ running time.

Fattest Augmenting Path

Fattest augmenting path.
- Finds augmenting path whose bottleneck capacity is maximum.
- Delivers most amount of flow to sink.
- Solve using Dijkstra-style (PFS) algorithm.

Choosing an Augmenting Path

Choosing an augmenting path.
- Any path will do ⇒ wide latitude in implementing Ford-Fulkerson.
- Generic priority first search.
- Some choices lead to good worst-case performance.
  - shortest augmenting path
  - fattest augmenting path
  - variation on a theme: PFS
- Average case not well understood.

Research challenges.
- Practice: solve max flow problems on real networks in linear time.
- Theory: prove it for worst-case networks.

History of Worst-Case Running Times

<table>
<thead>
<tr>
<th>Year</th>
<th>Discoverer</th>
<th>Method</th>
<th>Asymptotic Time</th>
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<tbody>
<tr>
<td>1951</td>
<td>Dantzig</td>
<td>Simplex</td>
<td>$E \cdot V^2 \cdot U$</td>
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<tr>
<td>1955</td>
<td>Ford, Fulkerson</td>
<td>Augmenting path</td>
<td>$E \cdot V \cdot U$</td>
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<tr>
<td>1970</td>
<td>Edmonds-Karp</td>
<td>Shortest path</td>
<td>$E^2 \cdot V$</td>
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<tr>
<td>1970</td>
<td>Edmonds-Karp</td>
<td>Max capacity</td>
<td>$E \cdot \log U \cdot (E + V \cdot \log V)$</td>
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<tr>
<td>1970</td>
<td>Dinitz</td>
<td>Improved shortest path</td>
<td>$E \cdot V^2$</td>
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<tr>
<td>1972</td>
<td>Edmonds-Karp, Dinitz</td>
<td>Capacity scaling</td>
<td>$E^2 \cdot \log U$</td>
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<tr>
<td>1973</td>
<td>Dinitz-Gabow</td>
<td>Improved capacity scaling</td>
<td>$E \cdot V \cdot \log U$</td>
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<td>1974</td>
<td>Karzanov</td>
<td>Preflow-push</td>
<td>$V^3$</td>
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<tr>
<td>1983</td>
<td>Sleator-Tarjan</td>
<td>Dynamic trees</td>
<td>$E \cdot V \cdot \log V$</td>
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<tr>
<td>1986</td>
<td>Goldberg-Tarjan</td>
<td>FIFO preflow-push</td>
<td>$E \cdot V \cdot \log (V^2 / E)$</td>
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</tbody>
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... ... ... ...

1997 | Goldberg-Rao        | Length function      | $E^{3/2} \cdot \log (V^2 / E) \cdot \log U$ |

† Arc capacities are between 1 and $U$. 
**An Application**

Jon placement.
- Companies make job offers.
- Students have job choices.

Can we fill every job?
Can we employ every student?

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<th>HP</th>
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**Bipartite Matching**

Input: undirected and bipartite graph $G$.
- Set of edges $M$ is a matching if each vertex appears at most once.
- Max matching: find a max cardinality matching.

Reduces to max flow.
- Create a directed graph $G'$.
- Direct all arcs from $L$ to $R$, and give infinite (or unit) capacity.
- Add source $s$, and unit capacity arcs from $s$ to each node in $L$.
- Add sink $t$, and unit capacity arcs from each node in $R$ to $t$. 
Claim. Matching in \( G \) of cardinality \( k \) induces flow in \( G' \) of value \( k \).

- Given matching \( M = \{ 1-B, 3-A, 4-E \} \) of cardinality 3.
- Consider flow \( f \) that sends 1 unit along each of 3 paths:
  \( s-1-B-t \quad s-3-A-t \quad s-4-E-t \).
- \( f \) is a flow, and has cardinality 3.

Reduction.

- Given an instance of bipartite matching.
- Transform it to a max flow problem.
- Solve max flow problem.
- Transform max flow solution to bipartite matching solution.

Issues.

- How expensive is transformation? \( O(E + V) \)
- Is it better to solve problem directly? \( O(EV^{1/2}) \) bipartite matching

Bottom line: max flow is an extremely rich problem-solving model.

- Many important practical problems reduce to max flow.
- We know good algorithms for solving max flow problems.