Balanced Trees

Splay trees

2-3-4 trees

Red-black trees

B-trees

Reference: Chapter 13, Algorithms in Java, 3rd Edition, Robert Sedgewick.

Symbol Table Review

Symbol table: key-value pair abstraction.
- Insert a value with specified key.
- Search for value given key.
- Delete value with given key.

Randomized BST.
- log N time per op (unless you get ridiculously unlucky).
- Store subtree count in each node.
- Generate random numbers for each insert/delete op.

This lecture.
- Splay trees.
- 2-3-4 trees.
- Red-black trees.
- B-trees.

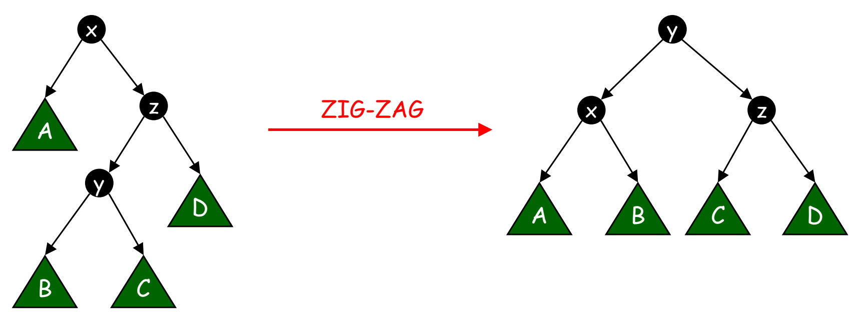
Splay Trees

Splay trees = self-adjusting BST.
- Tree automatically reorganizes itself after each op.
- After inserting x or searching for x, rotate x up to root using double rotations.
- Tree remains "balanced" without explicitly storing any balance information.

Amortized guarantee: any sequence of N ops takes O(N log N) time.
- Height of tree can be N.
- Individual op can take linear time.

Splay Trees

Splay.
- Check two links above current node.
- ZIG-ZAG: if orientations differ, same as root insertion.
- ZIG-ZIG: if orientations match, do top rotation first.



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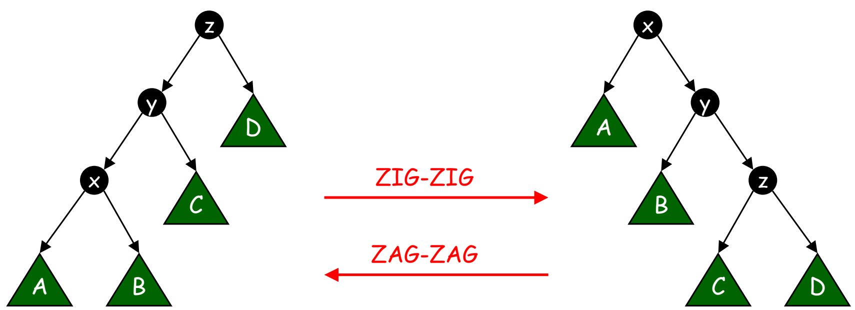

Splay Trees

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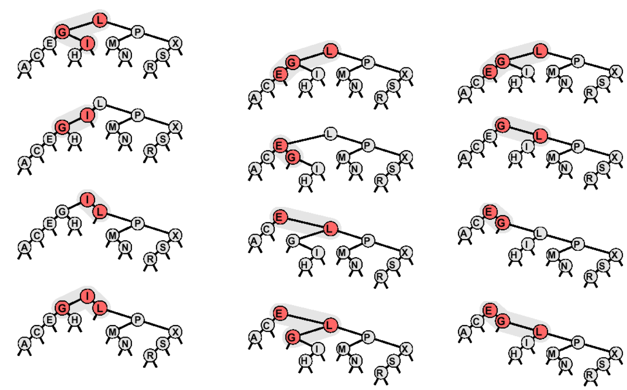


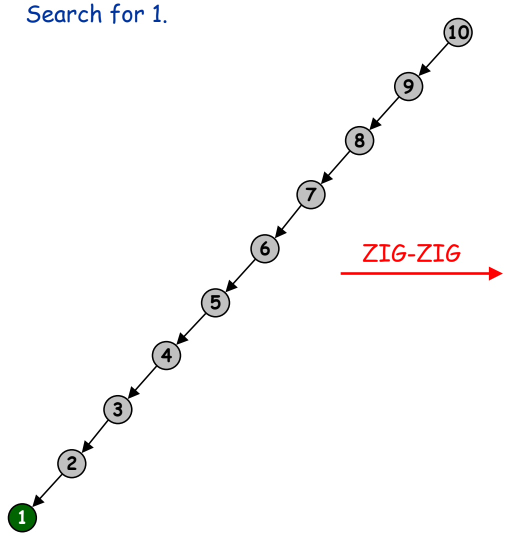
Root = Splay

Root Insertion

Splay Insertion

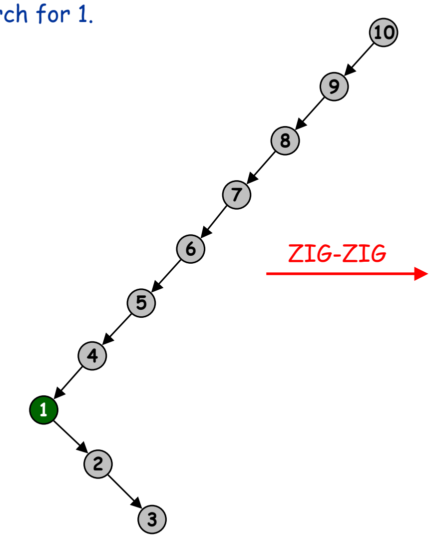
Splay Example

Search for 1.



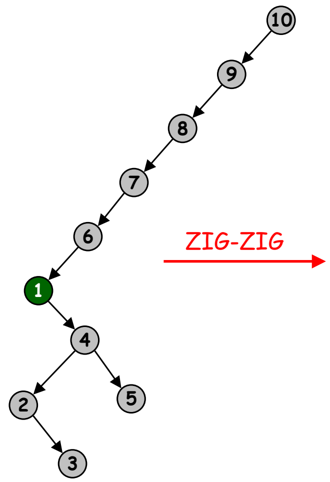
Splay Example

Search for 1.



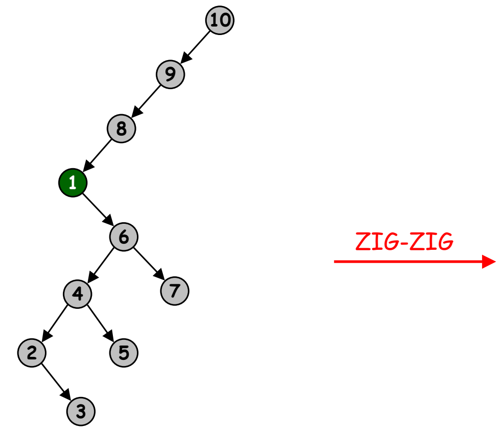
Splay Example

Search for 1.



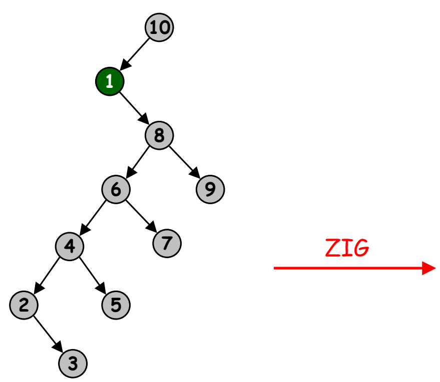
Splay Example

Search for 1.



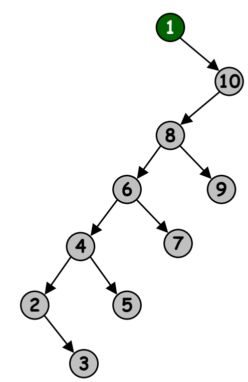
Splay Example

Search for 1.



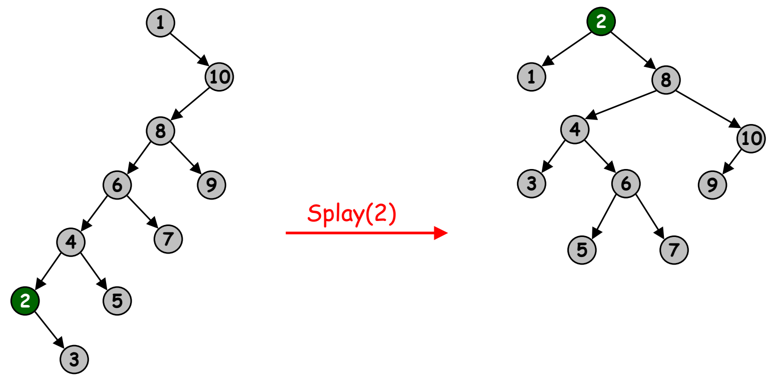
Splay Example

Search for 1.



Splay Example

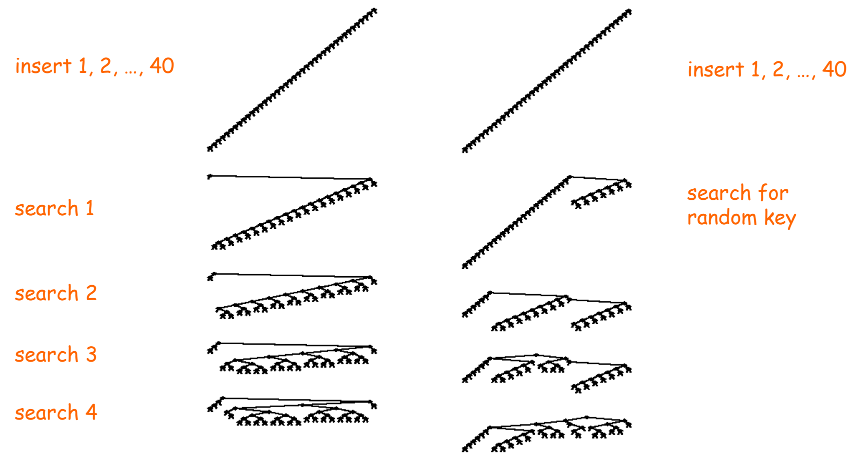
Search for 2.



Splay Trees

Intuition.

- Splay rotations halve search path.
- Reduces length of path for many other nodes in tree.



Symbol Table: Implementations Cost Summary

	Worst Case			Average Case		
Implementation	Search	Insert	Delete	Search	Insert	Delete
Sorted array	log N	N	N	log N	N	N
Unsorted list	N	1	1	N	1	1
Hashing	N	1	N	1*	1*	1*
BST	N	N	N	log N	log N	sqrt(N) †
Randomized BST	log N ‡	log N ‡	log N ‡	log N	log N	log N
Splay	log N §	log N §	log N §	log N §	log N §	log N §

* assumes we know location of node to be deleted
† if delete allowed, insert/search become sqrt(N)
‡ probabilistic guarantee
§ amortized guarantee

Splay: sequence of any N ops in O(N log N) time.
Ahead: Can we do all ops in log N time guaranteed?

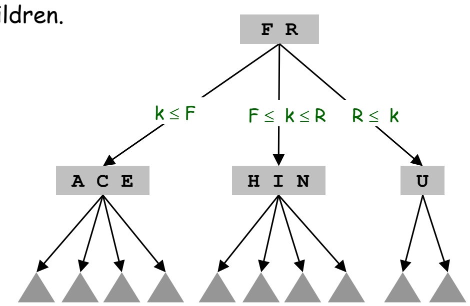
2-3-4 Trees

2-3-4 tree.

- Scheme to keep tree balanced.
- Generalize node to allow multiple keys.

Allow 1, 2, or 3 keys per node.

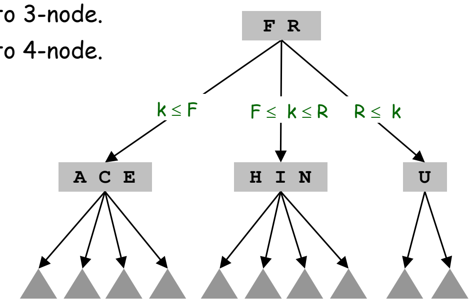
- 2-node: one key, two children.
- 3-node: two keys, three children.
- 4-node: three keys, four children.



2-3-4 Trees: Search and Insert

SEARCH.
- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

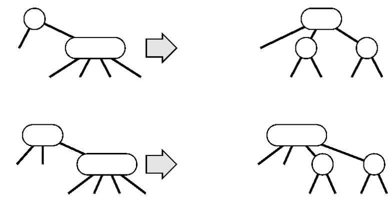
INSERT.
- Search to bottom for key.
- 2-node at bottom: convert to 3-node.
- 3-node at bottom: convert to 4-node.
- 4-node at bottom: ??



2-3-4 Trees: Splitting Four Nodes

Transform tree on the way DOWN.
- Ensure that last node is not a 4-node.

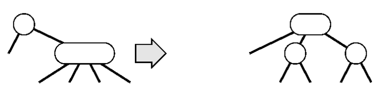
Local transformation to split 4-nodes:

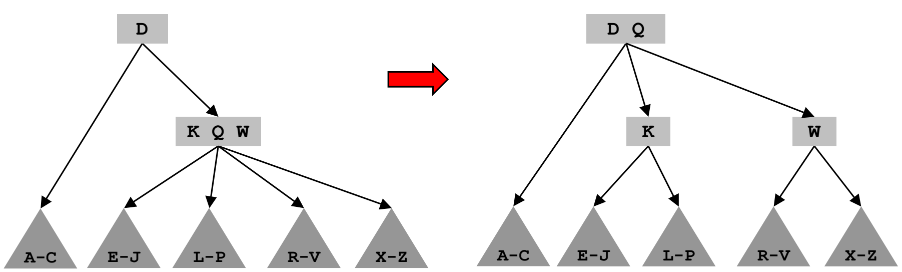


Invariant: current node is not a 4-node.
- One of two above transformations must apply at next node.
- Insertion at bottom is easy since it's not a 4-node.

2-3-4 Trees: Splitting a Four Node

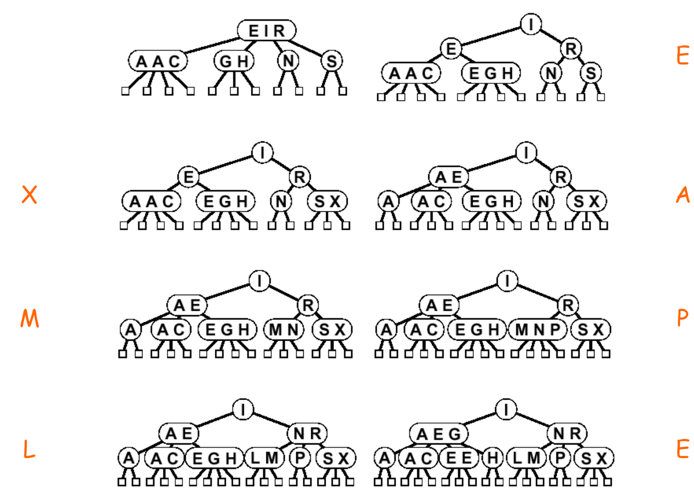
Splitting a four node: move middle key up.





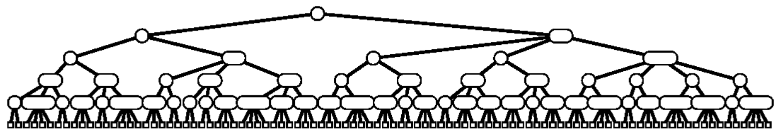
2-3-4 Trees

Tree grows up from the bottom.



Balance in 2-3-4 Trees

All paths from top to bottom have exactly the same length.



Tree height.

- Worst case: lg N all 2-nodes
- Best case: $\log_4 N = 1/2 \lg N$ all 4-nodes
- Between 10 and 20 for a million nodes.
- Between 15 and 30 for a billion nodes.

Comparison within nodes not accounted for.

2-3-4 Trees: Implementation?

Direct implementation complicated because of:

- Maintaining multiple node types.
- Implementation of `getChild`.
- Large number of cases for `split`.

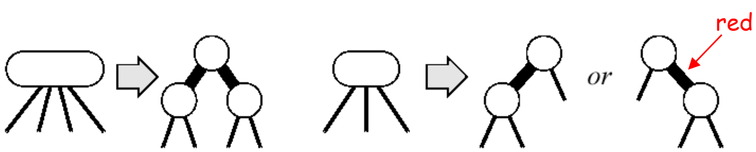
```
private Node insert(Node h, String key, Object value) {
   Node x = h;
   while (x != null) {
      x = x.getChild(key);
      if (x.is4Node()) x.split();
   }
   if      (x.is2Node()) x.make3Node(key, value);
   else if (x.is3Node()) x.make4Node(key, value);
}
```

Fantasy Code

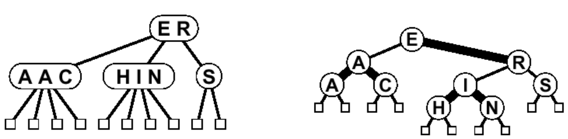
Red-Black Trees

Represent 2-3-4 trees as binary trees.

- Use "internal" edges for 3- and 4- nodes.



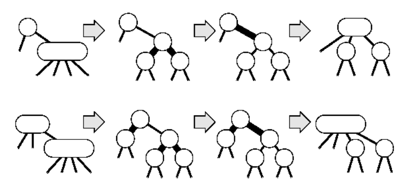
- Correspondence between 2-3-4 trees and red-black trees.



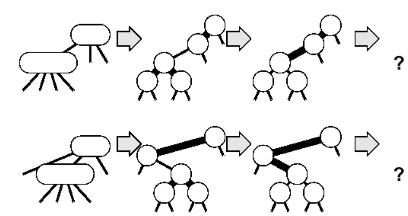
- Not 1-1 because 3-nodes swing either way.

Splitting Nodes in Red-Black Trees

Two cases are easy: switch colors.



Two cases require rotations.



do single rotation

do double rotation

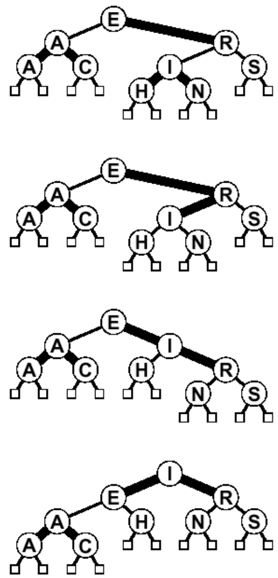
Red-Black Tree Node Split Example

inserting G

change colors

right rotate R →

left rotate E →



Red-Black Tree Construction

E

X

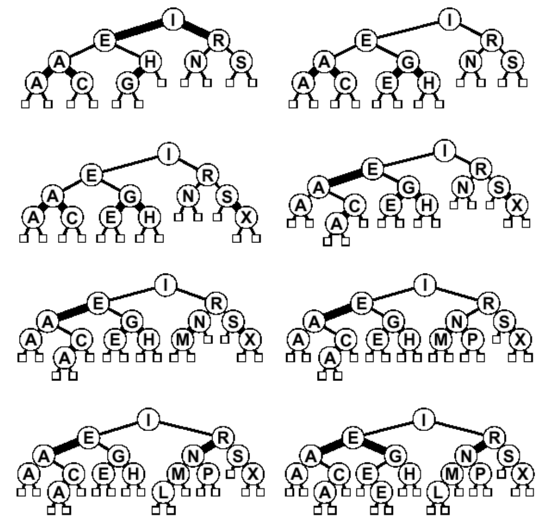
A

M

P

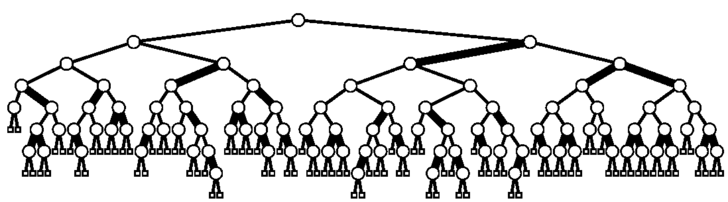
L

E



Balance in Red-Black Trees

Length of longest path is at most twice the length of shortest path.



Tree height.

- Worst case: 2 lg N.

Comparison within nodes ARE counted.

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Splay	log N §	log N §	log N §	log N §	log N §	log N §
Red-Black	log N	log N	log N	log N	log N	log N

* assumes hash map is random for all keys
† if delete allowed, insert/search become sqrt(N)
‡ probabilistic guarantee
§ amortized guarantee

Red-Black Trees in Practice

Red-black trees vs. splay trees.

- Fewer rotations than splay trees.
- One extra bit per node for color. ← possible to eliminate

Red-black trees vs. hashing.

- Hashing code is simpler and usually faster.
- Arithmetic to compute hash vs. comparison.
- Hashing performance guarantee is weaker.
- BSTs have more flexibility and can support wider range of ops.

Red-black trees are widely used as system symbol tables.

- Java: `TreeMap`, `TreeSet`.
- C++ STL: map, multimap, multiset.

Symbol Table: Java Libraries

Java has built-in library for red-black tree symbol table.

- `TreeMap` = red-black tree implementation.

```
import java.util.TreeMap;
public class TreeMapDemo {
   public static void main(String[] args) {
      TreeMap st = new TreeMap();
      st.put("www.cs.princeton.edu", "128.112.136.11");
      st.put("www.princeton.edu",    "128.112.128.15");
      st.put("www.simpsons.com",     "209.052.165.60");
      System.out.println(st.get("www.cs.princeton.edu"));
   }
}
```

Duplicate policy.

- Java `TreeMap` forbids two elements with the same key.
- Sedgewick implementations allows duplicate keys.

B-Trees

B-Tree generalize 2-3-4 trees by allowing up to M links per node.

- Split full nodes on the way down.

Main application: file systems.

- Reading a page into memory from disk is expensive.
- Accessing info on a page in memory is free.
- Goal: minimize # page accesses.
- Node size M = page size.

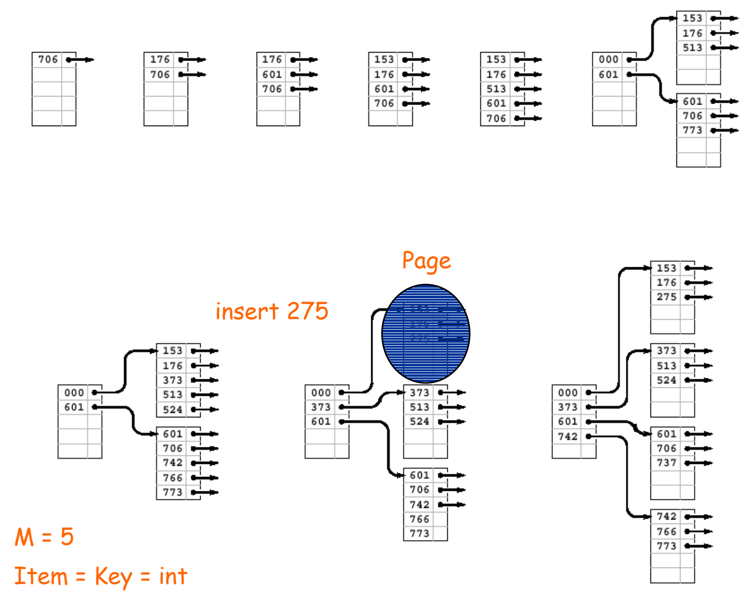
Space-time tradeoff.

- M large ⇒ only a few levels in tree.
- M small ⇒ less wasted space.
- Typical M = 1000, N < 1 trillion.

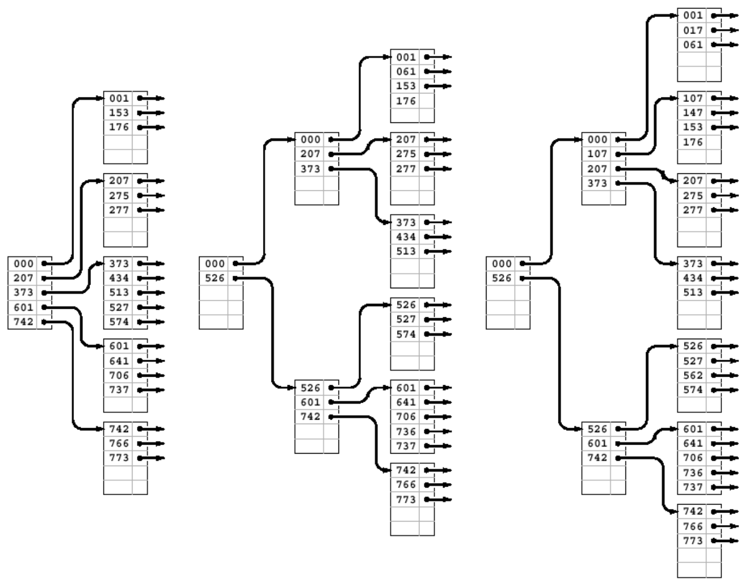
Bottom line: number of PAGE accesses is $\log_M N$ per op.

- 3 or 4 in practice!

B-Tree Example



B-Tree Example (cont)



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Splay	log N §	log N §	log N §	log N §	log N §	log N §
Red-Black	log N	log N	log N	log N	log N	log N
B-Tree	1	1	1	1	1	1

page accesses

B-Tree: Number of PAGE accesses is $\log_M N$ per op.

B-Tree in the Wild

File systems.
- Window HPFS (high performance file system).
- Mac HFS (hierarchical file system).
- Linux: ReiserFS, XFS, Ext3FS, JFS. ⬅ journaling

Databases.
- Most common index type in modern databases.
- ORACLE, DB2, INGRES, SQL, PostgreSQL, . . .

Variants.
- B trees: Bayer-McCreight (1972, Boeing)
- B+ trees: all data in external nodes.
- B* trees: keeps pages at least 2/3 full.
- R-trees for spatial searching: GIS, VLSI.

Summary

Goal: ST implementation with log N guarantee for all ops.
- Probabilistic: randomized BST.
- Amortized: splay tree, hashing. from re-doubling
- Worst-case: red-black tree.
- Algorithms are variations on a theme: rotations when inserting.

Abstraction extends to give search algorithms for huge files.
- B-tree.