CS 594: Approximation Algorithms
Homework 2
Due: Mon, Apr 14

1. Consider k unit vectors $v_1, v_2, \ldots, v_k$. Prove that

$$\max_{i \neq j}\{v_i \cdot v_j\} \geq \frac{-1}{k-1}$$

2. As with linear programs, semidefinite programs have duals. The dual of the MAX CUT SDP we looked at is

$$\frac{1}{2}\sum_{i<j} w_{ij} + \frac{1}{4}\min \sum_i \gamma_i$$

$$\text{subject to:} \qquad W + diag(\gamma) \text{ symmetric, positive semidefinite,}$$

where the matrix W is the symmetric matrix of the edge weights and the matrix $diag(\gamma)$ is the matrix with zeroes on the off-diagonal entries and γ_i as the ith entry on the diagonal. Show that the value of any feasible solution for this dual is an upper bound on the cost of any cut.

3. Conisder the following SDP formulation for min (s,t) cut.

$$\begin{aligned} & \min \sum_{(i,j)\in E} \frac{1 - v_i \cdot v_j}{2} \\ \forall i \qquad & v_i \cdot v_i = 1 \\ & v_s \cdot v_t = -1 \end{aligned}$$

(a) Construct an example to show that the SDP value could be much smaller than the value of the min (s,t) cut.

(b) Next, consider the strengthened formulation with triangle inequality constraints discussed in class. Prove that the value of the SDP equals the value of the min (s,t) cut in this case.

4. Semidefinite programming can also be used to give improved approximation algorithms for the MAX SAT problem. First we start with the MAX 2SAT problem, in which every clause has at most two literals.

(a) As in the case of MAX CUT, we'd like to express the MAX 2SAT problem as an "integer quadratic program" in which the only constraints are $y_i \in \{-1, 1\}$ and the objective function is quadratic in the y_i. Show that the MAX 2SAT problem can be expressed in this way. (Hint: it may help to introduce a variable y_0 which indicates whether the value -1 or $+1$ is "TRUE").

(b) Derive a 0.878 approximation algorithm for the MAX 2SAT problem.

(c) (Bonus problem) Use this 0.878-approximation algorithm for MAX 2SAT to derive a $\frac{3}{4} + \epsilon$-approximation algorithm for MAX SAT. How large an ϵ can you get ?