Problem Set No. 5
Collaboration is allowed

1. Suppose that a graph $G$ has a minimum spanning tree already computed. Describe an $O(n)$-time algorithm to update the minimum spanning tree if a new vertex and incident edges are added to $G$.

2. (Bi-directional search) The following version of Dijkstra’s algorithm finds the length of a shortest path from a vertex $s$ to a vertex $t$ by searching concurrently forward from $s$ and backward from $t$. For a vertex $v$, $d_s(v)$ is the tentative distance from $s$ to $v$ and $d_t(v)$ is the tentative distance from $v$ to $t$. The edge length function is $c(v,w)$, assumed nonnegative.

Initialization: Set $d_s(s) = d_t(t) = 0$. For $v \neq s$ set $d_s(v) = \infty$. For $v \neq t$ set $d_t(v) = \infty$. Set $r_s(s) = r_t(t) = true$. For $v \neq s$ set $r_s(v) = false$. For $v \neq t$ set $r_t(v) = false$. Set $Q_s = \{s\}$ and $Q_t = \{t\}$.

Main loop: While no vertex $v$ has $r_s(v) = r_t(v) = true$, perform one or the other of the following steps:

Forward from $s$: Delete from $Q_s$ a vertex $v$ with $d_s(v)$ minimum. Set $r_s(v) = true$. For each edge $(v,w)$, if $d_s(v) + c(v,w) < d_s(w)$, set $d_s(w) = d_s(v) + c(v,w)$ and add $w$ to $Q_s$ (if it is not already in $Q_s$).

Backward from $t$: Delete from $Q_t$ a vertex $v$ with $d_t(v)$ minimum. Set $r_t(v) = true$. For each edge $(u,v)$, if $d_t(u) > c(u,v) + d_t(v)$, set $d_t(u) = c(u,v) + d_t(v)$ and add $v$ to $Q_t$ (if it is not already in $Q_t$).

Describe how to extract the length of a shortest path from $s$ to $t$ when this algorithm terminates. Your extraction method should take $O(n)$ time. Prove the correctness of your answer.

3. A path cover of a directed graph $G = (V,E)$ is a set $P$ of vertex-disjoint paths such that every vertex in $V$ is included in exactly one path in $P$. Paths may start and end anywhere, and they may be of any length, including 0. A minimum path cover of $G$ is a path cover containing the fewest possible paths.


b. Does your algorithm work for directed graphs that contain cycles? Explain.