Linear Programming

**What is it?**
- Quintessential tool for optimal allocation of scarce resources, among a number of competing activities.
- Powerful and general problem-solving method.
  - shortest path, max flow, min cost flow, multicommodity flow, MST, matching, 2-person zero sum games

**Why significant?**
- Fast commercial solvers: CPLEX.
- Powerful modeling languages: AMPL, GAMS.
- Ranked among most important scientific advances of 20th century.
- Also a general tool for attacking NP-hard optimization problems.
- Dominates world of industry.
  - ex: Delta claims saving $100 million per year using LP

## Applications

- Agriculture. Diet problem.
- Computer science. Compiler register allocation, data mining.
- Electrical engineering. VLSI design, optimal clocking.
- Economics. Equilibrium theory, two-person zero-sum games.
- Environment. Water quality management.
- Finance. Portfolio optimization.
- Management. Hotel yield management.
- Marketing. Direct mail advertising.
- Manufacturing. Production line balancing, cutting stock.
- Physics. Ground states of 3-D Ising spin glasses.
- Telecommunication. Network design, Internet routing.
- Transportation. Airline crew assignment, vehicle routing.
- Sports. Scheduling ACC basketball, handicapping horse races.

## Brewery Problem: A Toy LP Example

**Small brewery produces ale and beer.**
- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

<table>
<thead>
<tr>
<th>Beverage</th>
<th>Corn (pounds)</th>
<th>Hops (ounces)</th>
<th>Malt (pounds)</th>
<th>Profit ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ale</td>
<td>5</td>
<td>4</td>
<td>35</td>
<td>13</td>
</tr>
<tr>
<td>Beer</td>
<td>15</td>
<td>4</td>
<td>20</td>
<td>23</td>
</tr>
</tbody>
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<th></th>
<th></th>
<th></th>
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<td>480</td>
<td>160</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beer</td>
<td></td>
<td></td>
<td>1190</td>
<td></td>
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**How can brewer maximize profits?**
- Devote all resources to ale: 34 barrels of ale  $\Rightarrow$ $442$.
- Devote all resources to beer: 32 barrels of beer  $\Rightarrow$ $736$.
- 7.5 barrels of ale, 29.5 barrels of beer  $\Rightarrow$ $776$.
- 12 barrels of ale, 28 barrels of beer  $\Rightarrow$ $800$. 

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Brewery Problem

**Objective Function**

\[
\text{Profit} = 13A + 23B
\]

\[
\text{s.t.} \quad 5A + 15B \leq 480
\]

\[
4A + 4B \leq 160
\]

\[
35A + 20B \leq 1190
\]

\[
A, B \geq 0
\]

**Geometry**

Brewery problem observation. Regardless of objective function coefficients, an optimal solution occurs at an extreme point.

**Feasible Region**

Constraints:

- \(4A + 4B \leq 160\)
- \(35A + 20B \leq 1190\)
- \(5A + 15B \leq 480\)

**Extreme Points**

- \((0, 32)\)
- \((12, 28)\)
- \((26, 14)\)
- \((0, 0)\)
- \((34, 0)\)
Standard Form LP

“Standard form” LP.
- Input data: rational numbers $c_j, b_i, a_{ij}$.
- Output: rational numbers $x_j$.
- $n = \#$ nonnegative variables, $m = \#$ constraints.
- Maximize linear objective function.
  - subject to linear inequalities

$$\begin{align*}
(P) \quad & \text{max} \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i \quad 1 \leq i \leq m \\
& x_j \geq 0 \quad 1 \leq j \leq n
\end{align*}$$

Linear. No $x^2$, $xy$, $\arccos(x)$, etc.

Programming. Planning (term predates computer programming).

Brewery Problem: Converting to Standard Form

Original input.

$$\begin{align*}
\text{max} \quad & 13A + 23B \\
\text{s.t.} \quad & 5A + 15B \leq 480 \\
& 4A + 4B \leq 160 \\
& 35A + 20B \leq 1190 \\
& A, B \geq 0
\end{align*}$$

Standard form.
- Add SLACK variable for each inequality.
- Now a 5-dimensional problem.

$$\begin{align*}
\text{max} \quad & 13A + 23B \\
\text{s.t.} \quad & 5A + 15B + S_H = 480 \\
& 4A + 4B + S_M = 160 \\
& 35A + 20B + S_C = 1190 \\
& A, B, S_H, S_M, S_C \geq 0
\end{align*}$$

Geometry

2-D geometry.
- Inequalities: halfplanes.
- Bounded feasible region: convex polygon.

Higher dimensional geometry.
- Inequalities: hyperplanes.
- Bounded feasible region: (convex) polytope.

Convex: if $y$ and $z$ are feasible solutions, then so is $(y+z)/2$.

Extreme point: feasible solution $x$ that can’t be written as $(y+z)/2$ for any two distinct feasible solutions $y$ and $z$.

Geometry

Extreme Point Property. If there exists an optimal solution to (P), then there exists one that is an extreme point.
- Only need to consider finitely many possible solutions.

Challenge. Number of extreme points can be exponential!
- Consider $n$-dimensional hypercube.

Greed. Local optima are global optima.
Simplex Algorithm

Simplex algorithm. (George Dantzig, 1947)
- Developed shortly after WWII in response to logistical problems.
- Used for 1948 Berlin airlift.

Generic algorithm.
- Start at some extreme point.
- Pivot from one extreme point to a neighboring one.
  - never decrease objective function
- Repeat until optimal.

How to implement?
- Use linear algebra.

Simplex Algorithm: Basis

Basis. Subset of $m$ of the $n$ variables.

Basic feasible solution (BFS). Set $n - m$ nonbasic variables to 0, solve for remaining $m$ variables.
- Solve $m$ equations in $m$ unknowns.
- If unique and feasible solution $\Rightarrow$ BFS.
- BFS corresponds to extreme point!
- Simplex only considers BFS.

Simplex Algorithm: Pivot 1

max $Z$ subject to
\[
\begin{align*}
13A + 23B & - Z = 0 \\
5A & + 15B + S_H = 480 \\
4A & + 4B + S_M = 160 \\
35A & + 20B + S_C = 1190 \\
A, B, S_H, S_M, S_C & \geq 0
\end{align*}
\]

Basis = \{S_H, S_M, S_C\}
A = B = 0
Z = 0
S_H = 480
S_M = 160
S_C = 1190

Substitute: $B = 1/15 (480 - 5A - S_H)$

max $Z$ subject to
\[
\begin{align*}
16 & A - \frac{23}{12} S_H & - Z = -736 \\
\frac{1}{3} & A + B + \frac{1}{15} S_H & = 32 \\
\frac{8}{3} & A - \frac{4}{15} S_H + S_M & = 32 \\
\frac{88}{3} & A - \frac{4}{3} S_H + S_C & = 550 \\
A, B, S_H, S_M, S_C & \geq 0
\end{align*}
\]

Basis = \{B, S_M, S_C\}
A = S_H = 0
Z = 736
B = 32
S_M = 32
S_C = 550

Why pivot on column 2?
- Each unit increase in $B$ increases objective value by $23$.
- Pivoting on column 1 also OK.

Why pivot on row 2?
- Ensures that RHS $\geq 0$ (and basic solution remains feasible).
- Minimum ratio rule: $\min \{ 480/15, 160/4, 1190/20 \}$.

Simplex Algorithm: Pivot 1

max $Z$ subject to
\[
\begin{align*}
13A + 23B & - Z = 0 \\
5A & + 15B + S_H = 480 \\
4A & + 4B + S_M = 160 \\
35A & + 20B + S_C = 1190 \\
A, B, S_H, S_M, S_C & \geq 0
\end{align*}
\]

Basis = \{S_H, S_M, S_C\}
A = B = 0
Z = 0
S_H = 480
S_M = 160
S_C = 1190

Infeasible
\{A, S_H, S_M\}
(12, 28)
\{A, B, S_M\}
(26, 14)
\{B, S_H, S_M\}
(0, 32)
\{B, S_H, S_C\}
(0, 0)
\{B, S_H, S_M\}
(34, 0)
\{S_H, S_M, S_C\}
(19.41, 25.53)
\{S_H, S_M, S_C\}
(12, 28)
Simplex Algorithm: Pivot 2

max Z subject to

\[
\begin{align*}
16 & \frac{8}{3} A - \frac{23}{12} S_H & - Z &= -736 \\
16 & \frac{A}{3} + B + \frac{3}{13} S_H &= 32 \\
& \frac{8}{3} S_H & + S_M &= 32 \\
& \frac{8}{3} A - \frac{3}{1} S_H + S_C &= 550 \\
A, B, S_H, S_M, S_C & \geq 0 \\
\end{align*}
\]

Substitute: \( A = \frac{3}{8} (32 + \frac{4}{15} S_H - S_M) \)

Basis = \{B, S_M, S_C\}

\[
\begin{align*}
A &= S_H = 0 \\
B &= 32 \\
S_M &= 32 \\
S_C &= 550 \\
Z &= 736 \\
A, B, S_H, S_M, S_C & \geq 0 \\
\end{align*}
\]

Simplex Algorithm: Optimality

When to stop pivoting?

- If all coefficients in top row are non-positive.

Why is resulting solution optimal?

- Any feasible solution satisfies system of equations in tableaux.
  - in particular: \( Z = 800 - S_H - 2 S_M \)
  - Thus, optimal objective value \( Z^* \leq 800 \) since \( S_H, S_M \geq 0 \).
  - Current BFS has value 800 \( \Rightarrow \) optimal.

Simplex Algorithm

Remarkable property. Simplex algorithm typically requires less than \( 2(m+n) \) pivots to attain optimality.

- No polynomial pivot rule known.
- Most pivot rules known to be exponential in worst-case.

Issues.

- Which neighboring extreme point?
- Cycling.
  - get stuck by cycling through different bases that all correspond to same extreme point
  - doesn’t occur in the wild
  - Bland’s least index rule \( \Rightarrow \) finite # of pivots
- Degeneracy.
  - new basis, same extreme point
  - “stalling” is common in practice

LP Duality: Economic Interpretation

Brewer’s problem: find optimal mix of beer and ale to maximize profits.

\[
(P) \quad \text{max} \quad 13A + 23B \\
\text{s.t.} \quad 5A + 15B \leq 480 \\
\quad \quad \quad 4A + 4B \leq 160 \\
\quad \quad \quad 35A + 20B \leq 1190 \\
\quad \quad \quad A, B \geq 0 \\
\]

\( A^* = 12 \)
\( B^* = 28 \)
\( \text{OPT} = 800 \)

Entrepreneur’s problem: Buy individual resources from brewer at minimum cost.

- \( C, H, M = \) unit price for corn, hops, malt.
- Brewer won’t agree to sell resources if \( 5C + 4H + 35M < 13 \).

\[
(D) \quad \text{min} \quad 480C + 160H + 1190M \\
\text{s.t.} \quad 5C + 4H + 35M \geq 13 \\
\quad \quad \quad 15C + 4H + 20M \geq 23 \\
\quad \quad \quad C, H, M \geq 0 \\
\]

\( C^* = 1 \)
\( H^* = 2 \)
\( M^* = 0 \)
\( \text{OPT} = 800 \)
**LP Duality**

Primal and dual LPs. Given rational numbers $a_{ij}$, $b_i$, $c_j$, find rational numbers $x_i$, $y_j$ that optimize (P) and (D).

\[
\begin{align*}
\text{(P)} \quad \text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} x_j \leq b_i \quad 1 \leq i \leq m \\
& \quad x_j \geq 0 \quad 1 \leq j \leq n
\end{align*}
\]

\[
\begin{align*}
\text{(D)} \quad \text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} y_j \geq c_j \quad 1 \leq j \leq n \\
& \quad y_i \geq 0 \quad 1 \leq i \leq m
\end{align*}
\]

Duality Theorem (Gale-Kuhn-Tucker 1951, Dantzig-von Neumann 1947).
If (P) and (D) have feasible solutions, then $\max = \min$.
- Special case: max-flow min-cut theorem.
- Sensitivity analysis.

**LP Duality: Economic Interpretation**

Sensitivity analysis.
- How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?
  - corn $1$, hops $2$, malt $0$.
- Suppose a new product "light beer" is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?
  - Breakeven: $2 \times (1) + 5 \times (2) + 24 \times (0) = 12$ / barrel.

How do I compute marginal prices (dual variables)?
- Simplex solves primal and dual simultaneously.
- Top row of final simplex tableaux provides optimal dual solution!

**History**

1939. Production, planning. (Kantorovich)
1947. Simplex algorithm. (Dantzig)
1950. Applications in many fields.
1979. Ellipsoid algorithm. (Khachian)
1984. Projective scaling algorithm. (Karmarkar)
1990. Interior point methods.

Current research.
- Approximation algorithms.
- Software for large scale optimization.
- Interior point variants.

**Ultimate Problem Solving Model**

Ultimate problem-solving model?
- Shortest path.
- Min cost flow.
- Linear programming.
- Semidefinite programming.
- TSP???
- (or any NP-complete problem)

Does $P = NP$?
- No universal problem-solving model exists unless $P = NP$. 
Perspective

LP is near the deep waters of NP-completeness.
- Solvable in polynomial time.
- Known for less than 25 years.

Integer linear programming.
- LP with integrality requirement.
- NP-hard.

An unsuspecting MBA student transitions from tractable LP to intractable ILP in a single mouse click.