Number Systems

- General form of a number in ***base*** ***b*** is

$$x = x_n b^n + x_{n-1} b^{n-1} + \dots + x_1 b^1 + x_0 b^0$$
$$+ x_{-1} b^{-1} + \dots + x_{-m} b^{-m}$$

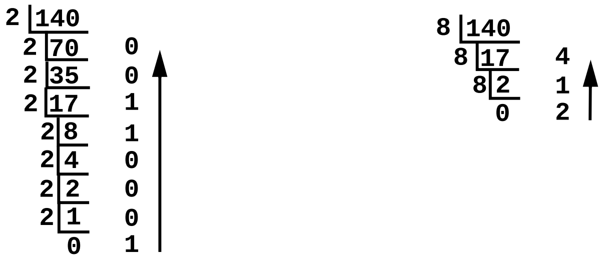
where x_i are the ***positional coefficients***

- Modern computers use binary arithmetic, i.e., base 2

$$\begin{aligned} 140_{10} &= 1 \times 10^2 + 4 \times 10^1 + 0 \times 10^0 \\ &= 1 \times 2^7 + 0 \times 2^6 + 0 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 \\ &= 10001100_2 \\ &= 2 \times 8^2 + 1 \times 8^1 + 4 \times 8^0 = 214_8 \\ &= 8 \times 16^1 + \mathrm{C} \times 16^0 = 8\mathrm{C}_{16} \end{aligned}$$

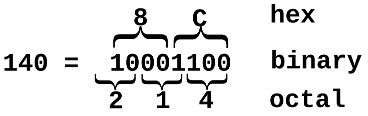
Conversions

- To convert from decimal to binary, divide by 2 repeatedly, read remainders up.



```
2|140                 8|140
  2|70   0              8|17   4
  2|35   0               8|2   1
  2|17   1                 0   2
   2|8   1
   2|4   0
   2|2   0
   2|1   0
     0   1
```

- Easier to convert to octal, then to binary



```
           8   C      hex
140 =  10001100       binary
        2  1  4       octal
```

Addition

- Addition in base ***b***

$$\begin{array}{r} x_n b^n + x_{n-1} b^{n-1} + x_{n-2} b^{n-2} + \ldots + x_1 b^1 + x_0 b^0 \\ + y_n b^n + y_{n-1} b^{n-1} + y_{n-2} b^{n-2} + \ldots + y_1 b^1 + y_0 b^0 \\ \hline z_{n+1} b^{n+1} + z_n b^n + z_{n-1} b^{n-1} + z_{n-2} b^{n-2} + \ldots + z_1 b^1 + z_0 b^0 \end{array}$$

where $S_i = x_i + y_i + C$, $C = S_{i-1} / b$, and $z_i = S_i \bmod b$ where $S_{-1} = 0$

- Addition in base 2:

```
  00101101
+ 10011001

  11000110
```

- the sum might have ***one*** more digit than the largest operand

Multiplication

- Multiplication in base 2: `00101101 * 10111001`

```
1 00101101
0  00000000
1   00101101
1    00101101
1     00101101
0      00000000
0       00000000
1        00101101

  010000010000101
```

- The product has about as many digits as the two operands combined, i.e.

$$\log(a \times b) = \log(a) + \log(b)$$

Machine Arithmetic

- Computers usually have a fixed number of binary digits (“bits”), e.g., 32 bits
- For example, using 6 bits, numbered 0 to 5 from the right

  largest number $111111_2 = 63_{10} = 2^6 - 1$

  smallest number $000000_2 = 0$

- What is 50 + 20?

```
   110010
 + 010100
   ------
  1000110
```

- The highest bit doesn’t fit, so we get $000110_2 = 6_{10}$
- Spilling over the lefthand side is ***overflow***

Sign Magnitude and One’s Complement

- ***Sign-magnitude*** notation:

  bit $n - 1$ is the sign; 0 for +, 1 for -

  bits $n - 2$ through 0 hold an unsigned number

  largest number $011111_2 = 31_{10} = 2^{6-1} - 1$

  smallest number $111111_2 = -31_{10} = -(2^{6-1} - 1)$

- Addition and subtraction are complicated when signs differ
- Sign-magnitude is rarely used
- ***One’s-complement*** notation: *-k = (2^n-1) - k = 11111...(n bits) - k*

  bit $n - 1$ is the sign; bits $n - 2$ through 0 hold an unsigned number

  bits $n - 2$ through 0 hold ***complement*** of negative numbers

  largest number $011111_2 = 31_{10} = 2^{6-1} - 1$

  smallest number $100000_2 = -31_{10} = -(2^{6-1} - 1)$

- Addition and subtraction are easy, but there are ***2*** representations for 0

Two's Complement

- ***Two's-complement*** notation: $-k = 2^n - k = \mathbf{(2^n\text{-}1) - k} + 1$

  bit $n-1$ is the sign; bits $n-2$ through 0 hold an unsigned number

  bits $n-2$ through 0 hold the ***complement*** of a negative number ***plus 1***

  largest number $011111_2 = 31_{10} = 2^{6-1} - 1$

  smallest number $100000_2 = -32_{10} = -2^{6-1}$; note ***asymmetry***

- To negate a 2's compl. number: first complement all the bits, then add 1

	start with	complement	increment	
+6	000110	111001	111010	-6
-6	111010	000101	000110	+6
+0	000000	111111	000000	-0
+1	000001	111110	111111	-1
+31	011111	100000	100001	-31
-31	100001	011110	011111	+31
-32	100000	011111	100000	-32

Two's Complement, Cont'd

- Adding 2's-complement numbers: ignore signs, add unsigned bit strings

```
    +20     010100          -20     101100
  + - 7   + 111001        + + 7   + 000111
    ———     ——————          ———     ——————
    +13     001101          -13     110011

    +20     010100          -20     101100
  + + 7   + 000111        + - 7   + 111001
    ———     ——————          ———     ——————
    +27     011011          -27     100101
```

- Signed overflow occurs if

  the carry ***into*** the sign bit differs from the carry ***out*** of the sign bit

```
    +20     010100          -20     101100
  + +17   + 010001        + -17   + 101111
    ———     ——————          ———     ——————
    -27     100101          +27     011011
```

- Same hardware for ***both*** unsigned and signed, but flags ***two*** conditions

  overflow signed overflow

  carry unsigned overflow

Sign Extension

- To convert from a small signed integer to a larger one, copy the sign bit

	+5	-5
4 bits	0101	1011
8 bits	00000101	11111011

- To convert a large signed integer to a smaller one: check trunced bits

	+5	-5	
8 bits	**00000101**	**11111011**	
4 bits	**0101**	**1011**	**OK!**

	+20	-20	
8 bits	00010100	11101100	
4 bits	0100	1100	Bad!

- Hardware does extension, but ***may not*** check for truncation; nor does C

```
short small = -50; long big = small;
printf("%d %d\n, small, big);          -50 -50

long big = 40000; short small = big;
printf("%d %d\n", small, big);         -25536 40000

char c = 255;
printf("%d\n", c);                     -1
```

Floating Point Numbers

- Floating point numbers are like scientific notation

$$1.386 \times 10^{6}$$

$$-3.0083 \times 10^{-14}$$

$$4.32 \times 10^{-8}$$

general form is

$$\pm m \times 10^{\pm p}$$

exponent

significand

- Significand restricted to range, e.g., $0 \le m < 1$, and fixed number of digits

- Floating point is approx. representation for infinitely many real numbers

$\boldsymbol{m} \times \beta^{\boldsymbol{k}}$	$\boldsymbol{m}$	is an $\boldsymbol{n}$-bit ***significand*** or ***fraction***
	β	is the ***base*** (usually 2)
	$\boldsymbol{k}$	is the ***exponent***

e.g. for base 2

$$0.100011 \times 2^{6} = (1 \times 2^{-1} + 0 \times 2^{-2} + 0 \times 2^{-3} + 0 \times 2^{-4} + 1 \times 2^{-5} + 1 \times 2^{-6}) \times 2^{6}$$

Floating Point Numbers, cont'd

- ***Normalized*** floating point numbers make the representation unique

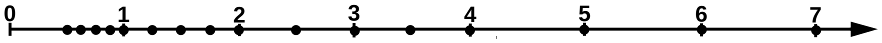
  most significant digit is nonzero, e.g., $0.00486 \times 10^{1} \Rightarrow 0.486 \times 10^{-1}$

  for floating point numbers, $\beta^{n-1} \leq m < \beta^{n}$ or $1/\beta \leq |m| < 1$

  i.e., when $\beta = 2$, most significant bit of $\boldsymbol{m}$ is 1

- Example: $n = 3, \beta = 2, -1 \leq k \leq 2$

		k			
		-1	**0**	**1**	**2**
m	**1.00**	**.5**	**1.**	**2.**	**4.**
	1.01	**.625**	**1.25**	**2.5**	**5.**
	1.10	**.75**	**1.5**	**3.**	**6.**
	1.11	**.875**	**1.75**	**3.5**	**7.**
δ		**.125**	**.25**	**.5**	**1.**



- What about 0.0? Use reserved values of $\boldsymbol{k}$, e.g.,

  $1.00_2 \times 2^{-2}$ for 0.0, $1.11_2 \times 2^{5}$ for ∞

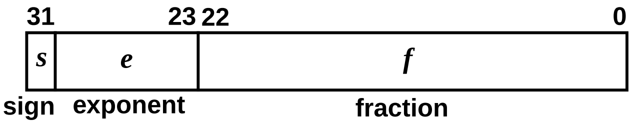
IEEE Floating Point

- IEEE format uses a ***hidden bit*** to increase precision by 1 bit

  all ***normalized*** floating point numbers have the form $1.f \times 2^{e}$,

  so ***assume*** the leading 1 and omit it

- Single precision (`float`) format



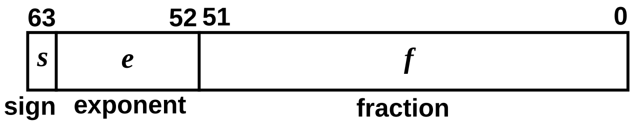
$-126 \leq e \leq 127$, $\boldsymbol{bias} = 127, 0 \leq f < 2^{23}$

- Values `1.1754943508222875e-38` to `3.4028234663852886000e+38`

$\boldsymbol{k} = \boldsymbol{e} - 127$	$\boldsymbol{f}$	**f. p. number**
$-126 \leq k \leq 127$	$0 \leq f < 2^{23}$	$\pm 1.f \times 2^{k}$
128	0	$\pm\infty$
128	$\neq 0$	**NaN (signaling/quiet)**
-127	0	± 0.0
-127	$\neq 0$	$\pm 0.f \times 2^{-126}$ **(denormalized)**

IEEE Floating Point, cont'd

- Double precision (**`double`**) format



$-1022 \le e \le 1023$, $\boldsymbol{bias} = 1023$, $0 \le f < 2^{52}$

- Values: **`2.2250738585072014e-308`** to **`1.7976931348623157e+308`**

$\boldsymbol{k} = \boldsymbol{e} - 1023$	$\boldsymbol{f}$	**f. p. number**
$-1022 \le k \le 1023$	$0 \le f < 2^{52}$	$\pm 1.f \times 2^{k}$
1024	0	$\pm\infty$
1024	$\ne 0$	**NaN (signaling/quiet)**
-1023	0	± 0.0
-1023	$\ne 0$	$\pm 0.f \times 2^{-1022}$ **(denormalized)**

- Biased exponents in the most-significant bits are useful because

  integer compare instructions can be used to compare floating point values

  a bit string of 0's represents the value 0.0