Homework 1: due Oct 26

Problem 1 Prove that in a binomial tree with n nodes

$$\sum_v 2^{\mathrm{rank}(v)/2} \leq cn,$$

for some constant $c > 0$, where the sum runs over all the nodes of the tree.

Problem 2 Let T be a rooted tree, where each internal node has exactly two children (but the tree need not be balanced). Given a node v of T, let $N(v)$ denote the number of leaves in the subtree rooted at v. Prove that

$$\sum_v \min\{\log N(v_1), \log N(v_2)\} = O(n),$$

where the sum runs over all internal nodes v, with v_1 (resp. v_2) denoting the left (resp. right) child of v.

Problem 3 We want to prove that the entropy of the MST problem is at most linear. Given a connected, undirected graph G with n vertices and m edges (no loops or multiple edges), let $E(G)$ denote the logarithm (in base 2) of the number of distinct spanning trees.

1. Replace each edge of G by two directed edges (creating a 2-edge cycle) and form the $2m$-by-n matrix D, where each column (resp. row) corresponds to a vertex (resp. edge): The row associated with the directed edge (i, j) has -1 in column i, 1 in column j, and 0 elsewhere.
2. Prove that D has rank $n - 1$ if G is connected.
3. Prove that D is unimodular, ie, each square submatrix of D has determinant $0, 1$, or -1.
4. Let $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$ be the eigenvalues of $L = D^T D$. Prove that $\mu_1 = 0$.
5. Using the fact (don't prove it) that the number of spanning trees is equal to $\mu_2 \cdots \mu_n / n$, show that $E(G) = O(m)$. You may restrict yourself to the case of a regular matrix (ie, one derived from a graph

where each vertex has the same number of incident edges) and use the fact that the spectral norm of a matrix $A = (a_{ij})$ is at most

$$\Big(\max_j \sum_i |a_{ij}|\Big)^{1/2} \Big(\max_i \sum_j |a_{ij}|\Big)^{1/2}.$$

Problem 4 A set F of functions from $\{0,1\}^m$ to $\{0,1\}^l$ ($l < m$) is said to form a family of universal hash functions if: given any $x \neq y \in \{0,1\}^m$, a random f chosen uniformly in F is such that

$$\mathrm{Prob}[f(x) = f(y)] \leq 2^{-l} + 2^{-m}.$$

1. Prove that the set F of all linear functions from $\{0,1\}^m$ to $\{0,1\}^l$ forms a family of universal hash functions.

2. Fix a prime p with binary representation of length $m+1$, and let F consist of all functions

$$x \mapsto (ax + b \pmod p) \pmod{2^l},$$

for all integers $0 \leq a, b < p$. Prove that F is a family of universal hash functions.