Challenges of splines

Challenge of splines:

- Continuity (smoothness at joints)

Continuity C^k indicates adjacent curves have the same kth derivative at their joints

C^0 Continuity

Adjacent curves share …

- Same endpoints: $Q_i(1) = Q_{i+1}(0)$

C^0 Continuity

Conditions for uniform cubic Bezier …

- Same endpoints: $V_3 = W_0$

C^1 Continuity

Adjacent curves share …

- Same endpoints: $Q_i(1) = Q_{i+1}(0)$
- Same derivatives: $Q_i'(1) = Q_{i+1}'(0)$

C^1 Continuity

Conditions for uniform cubic Bezier …

- Same endpoints: $V_3 = W_0$
- Same derivatives: $(V_3 - V_2) = (W_1 - W_0)$

C^2 Continuity

Adjacent curves share …

- Same endpoints: $Q_i(1) = Q_{i+1}(0)$
- Same derivatives: $Q_i'(1) = Q_{i+1}'(0)$
- Same second derivatives: $Q_i''(1) = Q_{i+1}''(0)$

C^2 Continuity

Conditions for uniform cubic Bezier …

- Same endpoints: $V_3 = W_0$
- Same derivatives: $(V_3 - V_2) = (W_1 - W_0)$
- Same second derivatives:
  $(V_3 - V_2) - (V_2 - V_1) = (W_2 - W_1) - (W_1 - W_0)$

Creating continuous cubic splines

Question: how should we choose Bezier control points to construct a smooth spline through a set of points?

Goals:

- Interpolate points
- Convex hull property
- C^2 continuity
- Local control

Spline constructions

C^2 interpolating splines

Catmull-Rom splines

B-splines

C^2 Interpolating splines

Bezier control points are chosen so that …

- Control points are interpolated
- Adjacent curves meet with C^2 continuity

C^2 Interpolating splines

Properties:

- Interpolate control points
- C^2 continuity
- No convex hull property
- No local control

Spline constructions

C^2 interpolating splines

Catmull-Rom splines

B-splines

Catmull-Rom splines

Bezier control points are chosen so that ...

- Points are interpolated
- Adjacent curves meet with C^1 continuity **(not C^2)**
- Local control

Catmull-Rom splines

Derivation:

- Three conditions for each point P_i …
  - – Last curve ends at P_i
  - – Next curve begins at P_i
  - – Tangents of two curves are equal at P_i

One degree of freedom for each point

Catmull-Rom Splines

Matrix form for uniform cubic Catmull-Rom splines:

$$Q(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 3/2 & -3/2 & 1/2 \\ 1 & -5/2 & 2 & -1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$

Catmull-Rom splines

Properties:

- Interpolate points
- C^1 continuity
- Local control
- No convex hull property

Spline constructions

C2 interpolating splines

Catmull-Rom splines

B-splines

Uniform Cubic B-Splines

Choose Bezier control points so that …

- C^2 continuity
- Local control
- Points not necessarily interpolated

Uniform Cubic B-Splines

Derivation:

- Three continuity conditions for each joint J_i …
  - – Position of two curves are equal at J_i
  - – Derivatives of two curves are equal at J_i
  - – Second derivatives of two curves are equal at J_i
- Also, local control implies …
  - – Each joint is affected by small set of (4) points

Uniform Cubic B-Splines

Fifteen continuity constraints:

$0 = b_{-0}(0)$	$0 = b_{-0}'(0)$	$0 = b_{-0}''(0)$
$b_{-0}(1) = b_{-1}(0)$	$b_{-0}'(1) = b_{-1}'(0)$	$b_{-0}''(1) = b_{-1}''(0)$
$b_{-1}(1) = b_{-2}(0)$	$b_{-1}'(1) = b_{-2}'(0)$	$b_{-1}''(1) = b_{-2}''(0)$
$b_{-2}(1) = b_{-3}(0)$	$b_{-2}'(1) = b_{-3}'(0)$	$b_{-2}''(1) = b_{-3}''(0)$
$b_{-3}(1) = 0$	$b_{-3}'(1) = 0$	$b_{-3}''(1) = 0$

One more convenient constraint:

$$b_{-0}(0) + b_{-1}(0) + b_{-2}(0) + b_{-3}(0) = 1$$

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Uniform Cubic B-Splines

Solving the system of equations:

$$
\begin{aligned}
b_{-3}(u) &= -\tfrac{1}{6}u^3 + \tfrac{1}{2}u^2 - \tfrac{1}{2}u + \tfrac{1}{6} \\
b_{-2}(u) &= \tfrac{1}{2}u^3 - u^2 + \tfrac{2}{3} \\
b_{-1}(u) &= -\tfrac{1}{2}u^3 + \tfrac{1}{2}u^2 + \tfrac{1}{2}u + \tfrac{1}{6} \\
b_{-0}(u) &= \tfrac{1}{6}u^3
\end{aligned}
$$

Uniform Cubic B-Splines

Matrix form for uniform cubic B-spline:

$$
Q(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}
\begin{bmatrix}
-\tfrac{1}{6} & \tfrac{1}{2} & -\tfrac{1}{2} & \tfrac{1}{6} \\
\tfrac{1}{2} & -1 & \tfrac{1}{2} & 0 \\
-\tfrac{1}{2} & 0 & \tfrac{1}{2} & 0 \\
\tfrac{1}{6} & \tfrac{2}{3} & \tfrac{1}{6} & 0
\end{bmatrix}
\begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}
$$

Uniform Cubic B-Splines

Properties:

- C^2 continuity
- Local control
- Approximating (points not interpolated)
- Convex hull property

Summary

Goals:

- Interpolate points
- Convex hull property
- C^2 continuity
- Local control

Results:

- No uniform cubic spline curve satisfies all goals
  - Too few degrees of freedom in cubic polynomials
- Choose spline construction method for each application