Solutions to Midterm Exam

Problem 1 (a) For even \( n = 2m \) where \( m > 0 \) is an integer, \( f(n) = \sum_{1 \leq i \leq m} (-2i - 1 + (2i)) = m = (-1)^n \lfloor n/2 \rfloor \).

For odd \( n = 2m + 1 \) where \( m \geq 0 \) is an integer, \( f(n) = -1 + \sum_{1 \leq i \leq m} (2i - (2i + 1)) = -(m + 1) = (-1)^n \lfloor n/2 \rfloor \).

(b) From (5.17) in the textbook (page 148), we have

\[
\frac{1}{(1-x)^n} = \sum_{k \geq 0} \binom{n+k-1}{k} x^k.
\]

Thus,

\[
\sum_{k \geq 0} \frac{1}{4^k} \binom{n+k-1}{k} = \frac{1}{(1-1/4)^n} = (4/3)^n.
\]

Problem 2 From (5.14) in the textbook (page 136), we have

\[
\sum_{r \leq i \leq m} \binom{i}{r} = \binom{m+1}{r+1}.
\]

It follows that

\[
\sum_{0 \leq i \leq m} i^2 = \sum_{1 \leq i \leq m} \left( \binom{i}{2} + i \right)
= 2 \sum_{2 \leq i \leq m} \binom{i}{2} + \sum_{1 \leq i \leq m} \binom{i}{1}
= 2 \binom{m+1}{3} + \binom{m+1}{2}.
\]

Now for each \( 0 \leq i \leq \sqrt{n} \), there are exactly \( n + 1 - i^2 \) elements \((i,j) \in S_n\). Thus,

\[
h(n) = \sum_{0 \leq i \leq \sqrt{n}} (n - i^2 + 1)
= (n + 1)(\sqrt{n} + 1) - \sum_{0 \leq i \leq \sqrt{n}} i^2
= (n + 1)(\sqrt{n} + 1) - 2 \left( \frac{\sqrt{n} + 1}{3} \right) - \left( \frac{\sqrt{n} + 1}{2} \right).
\]
Problem 3

(a) The probability space is \( \Omega = (A, p) \), where \( A = \{H, T\}^{2n} \) and \( p(a) = 1/4^n \) for all \( a \in A \). The event \( E_1 \) is the set of all \( a = (a_1, a_2, \cdots, a_n, b_1, \cdots, b_n) \) such that the number of \( H \)'s among \( a_i \)'s is equal to the number of \( H \)'s among the \( b_j \)'s. Note that there are exactly \( \binom{n}{k} \binom{n}{n-k} \) points \( a \in E \) such that there are \( k \) \( H \)'s among \( a_i \)'s and also among \( b_j \)'s. Thus,

\[
    s(n) = \sum_{a \in E_1} p(a) \\
    = \frac{1}{4^n} |E_1| \\
    = \frac{1}{4^n} \sum_{0 \leq k \leq n} \binom{n}{k} \binom{n}{k} \\
    = \frac{1}{4^n} \binom{2n}{n},
\]

where in the last step we have used (5.11) in the textbook (page 133).

(b) The event \( E_2 \) is the set of all \( a = (a_1, a_2, \cdots, a_n, b_1, \cdots, b_n) \) such that the number of \( H \)'s among \( a_i \)'s is equal to \( 1 \) plus the number of \( H \)'s among the \( b_j \)'s. Note that there are exactly \( \binom{n}{k} \binom{n}{n-k-1} \) points \( a \in E \) such that there are \( k \) \( H \)'s among \( a_i \)'s and \( k-1 \) \( H \)'s among \( b_j \)'s. Thus,

\[
    t(n) = \sum_{a \in E_2} p(a) \\
    = \frac{1}{4^n} |E_2| \\
    = \frac{1}{4^n} \sum_{1 \leq k \leq n} \binom{n}{k} \binom{n}{k-1} \\
    = \frac{1}{4^n} \sum_{1 \leq k \leq n} \binom{n}{k} \binom{n}{n+1-k}.
\]

However,

\[
    \sum_{1 \leq k \leq n} \binom{n}{k} \binom{n}{n+1-k} = \binom{2n}{n+1},
\]

since both sides are equal to the number of \((n+1)\)-combinations of \(\{1, 2, \cdots, 2n\}\). (Note that \( \binom{n}{k} \binom{n}{n+1-k} \) is the number of such combinations in which \( k \) of the elements come from \(\{1, 2, \cdots, n\}\), and \( n+1-k \) come from \(\{n+1, n+2, \cdots, 2n\}\). Thus,

\[
    t(n) = \frac{1}{4^n} \binom{2n}{n+1}.
\]