Midterm Exam

Do all three problems. To get full credits, your answers must be given as closed-form expressions of $n$. In your derivations, you may use (no proof needed, just refer to the page number) any formula given in the textbook.

In the problems below, $n$ is always a positive integer.

**Problem 1** [20 pts] (a) Let $f(n) = \sum_{0 \leq k \leq n} (-1)^k k$. Prove that $f(n) = (-1)^n \lfloor n/2 \rfloor$.
(b) Let $g(n) = \sum_{k \geq 0} \frac{1}{4^k} \binom{n + k - 1}{k}$. Determine $g(n)$.

**Problem 2** [20 pts] Let $n$ be any integer of the form $m^2$ for some positive integer $m$. Let $S_n$ be the set of all pairs $(i, j)$ such that (i) $i, j$ are non-negative integers, and (ii) $i^2 + j \leq n$. Let $h(n) = |S_n|$. Determine $h(n)$.

**Problem 3** [20 pts] Alice and Bob each tosses an unbiased coin $n$ times. Let $X$ and $Y$ be the random variables corresponding to the number of HEADs in Alice’ and Bob’s results.
(a) Let $E_1$ denote the event that $X = Y$, and let $s(n) = \Pr\{E_1\}$. Determine $s(n)$.
(b) Let $E_2$ denote the event that $X = Y + 1$, and let $t(n) = \Pr\{E_2\}$. Determine $t(n)$.

*Remarks* Note that $s(1) = 1/2, s(2) = 3/8, t(1) = 1/4, t(2) = 1/4$. You may want to check your answers for $n = 1, 2$ against these values.