Solving Recurrence with Generating Functions

The first problem is to solve the recurrence relation system $a_0 = 1$, and $a_n = a_{n-1} + n$ for $n \geq 1$.

Let $A(x) = \sum_{n\geq0} a_n x^n$. Multiply both side of the recurrence by $x^n$ and sum over $n \geq 1$. This gives

$$\sum_{n\geq1} a_n x^n = x \sum_{n\geq1} a_{n-1} x^{n-1} + \sum_{n\geq1} n x^n.$$  

Note that

$$\sum_{n\geq1} n x^n = \sum_{n\geq0} n x^n$$

$$= x \frac{d}{dx} \left( \sum_{n\geq0} x^n \right)$$

$$= x \frac{d}{dx} \frac{1}{1-x}$$

$$= \frac{x}{(1-x)^2}.$$  

Thus, in term of $A(x)$, we obtain

$$A(x) - 1 = xA(x) + \frac{x}{(1-x)^2}.$$  

Rearranging terms, we get

$$(1-x)A(x) = 1 + \frac{x}{(1-x)^2}.$$  

Hence,

$$A(x) = \frac{1}{1-x} + x(1-x)^{-3}.$$  

We can now get $a_n$ by expanding $A(x)$ as a series

$$A(x) = \sum_{n\geq0} x^n + x \sum_{n\geq0} \binom{-3}{n} (-1)^n x^n.$$  

This gives, for all $n \geq 0$,

$$a_n = 1 + \binom{-3}{n-1} (-1)^{n-1}.$$  

Now
\[
\binom{-3}{k} = \frac{(-3)(-4) \cdots (-3-k+1)}{k!}
\]
\[
= \frac{(-1)^k(k+2)(k+1)}{2}
\]
\[
= (-1)^k\binom{k+2}{2}.
\]

Thus,
\[
a_n = 1 + \binom{n+1}{2}.
\]

This is the same answer as we obtained earlier by different means.

The next problem for solution is the Rabbit Island problem. Before studying it, let us note the following identity, valid for any distinct numbers \(b\) and \(c\):
\[
\frac{1}{(1-bx)(1-cx)} = \frac{1}{b-c} \left( \frac{b}{1-bx} - \frac{c}{1-cx} \right).
\]

It can be directly verified by taking common denominators of the terms on the right-hand-side, and simplifying the expression. A more systematic way to do this is to solve the system of equations for variables \(\lambda, \mu,\)
\[
\lambda + \mu = 1, \quad \lambda b + \mu c = 0.
\]

The solution satisfies the equation
\[
1 = \lambda(1-bx) + \mu(1-cx),
\]
and gives
\[
\frac{1}{(1-bx)(1-cx)} = \frac{\lambda(1-bx) + \mu(1-cx)}{(1-bx)(1-cx)}
\]
\[
= \frac{\lambda}{1-cx} + \frac{\mu}{1-bx}.
\]

This is a special case of the partial fraction decomposition. You might find it challenging to extend the discussion to show that, if \(b, c, d\) are distinct,
\[
\frac{1}{(1-bx)(1-cx)(1-dx)} = \frac{\lambda}{1-dx} + \frac{\mu}{1-cx} + \frac{\gamma}{1-bx},
\]
with some appropriate choice of \(\lambda, \mu, \gamma\).

In the Rabbit Island problem, we need to solve the recurrence \(a_0 = 0, a_1 = 1,\) and \(a_n = a_{n-1} + a_{n-2}\) for \(n \geq 2\). Let \(A(x) = \sum_{n \geq 0} a_n x^n\). As in the previous problem, let us multiply the recurrence by \(x^n\) and sum over \(n \geq 2\). This gives
\[
\sum_{n \geq 2} a_n x^n = x \sum_{n \geq 2} a_{n-1} x^{n-1} + x^2 \sum_{n \geq 2} a_{n-2} x^{n-2}.
\]
In terms of \( A(x) \), we have \( A(x) - a_0 - a_1 x = x(A(x) - a_0) + x^2 A(x) \). This leads to

\[
A(x) = \frac{x}{1 - x - x^2}.
\]  

(2)

It remains to expand \( A(x) \) into a power series, so that we can identify \( a_n \).

Now note that

\[
1 - x - x^2 = 1 - x + \frac{x^2}{4} - \frac{5x^2}{4}
\]

\[
= \left( 1 - \frac{x}{2} \right)^2 - \left( \frac{\sqrt{5} x}{2} \right)^2
\]

\[
= \left( 1 - \frac{x}{2} - \frac{\sqrt{5} x}{2} \right) \left( 1 - \frac{x}{2} + \frac{\sqrt{5} x}{2} \right)
\]

\[
= (1 - bx)(1 - cx),
\]

where \( b = (1 + \sqrt{5})/2 \) and \( c = (1 - \sqrt{5})/2 \). Using (1) and (2), we can expand \( A(x) \) as

\[
A(x) = \frac{x}{(1 - bx)(1 - cx)}
\]

\[
= \frac{bx}{b - c} \frac{1}{1 - bx} \frac{cx}{b - c} \frac{1}{1 - cx}
\]

\[
= \frac{bx}{b - c} \sum_{n \geq 0} (bx)^n - \frac{cx}{b - c} \sum_{n \geq 0} (cx)^n
\]

\[
= \frac{1}{\sqrt{5}} \sum_{n \geq 0} (b^{n+1} - c^{n+1})x^{n+1}
\]

\[
= \frac{1}{\sqrt{5}} \sum_{n \geq 1} (b^n - c^n)x^n.
\]

Thus, for all \( n \geq 1 \), we have

\[
a_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right).
\]

For \( n = 1 \), this formula gives \( a_1 = 1 \), as was to be expected. Incidentally, the above formula also gives the correct value \( a_0 = 0 \).

The numbers \( a_n \) are called Fibonacci numbers, and often denoted by \( F_n \). Note that \( b = 1.6 \cdots \) and \( c = -0.6 \cdots \). Thus, \( c^n \) is numerically a very small number, while \( b^n \) is large. For reasonably large \( n \), say \( n > 10 \), \( F_n \) can be obtained by evaluating \( \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n \), and rounding it to the closest integer.

The third problem we tackle is the recurrence \( a_0 = 0 \), \( a_1 = 1 \), and \( a_n = \sum_{1 \leq i \leq n-1} a_i a_{n-i} \) for \( n \geq 2 \). The quantity \( a_n \) is the number of ways to parenthesize an expression \( y_1 + y_2 + \cdots + y_n \).
Let $A(x) = \sum_{n \geq 0} a_n x^n = \sum_{n \geq 1} a_n x^{n-1}$. The recurrence relation gives

\[
\sum_{n \geq 2} a_n x^n = \sum_{n \geq 1} \sum_{i \leq n-1} a_i x^i a_{n-i} x^{n-i} = (\sum_{i \geq 1} a_i x^i)(\sum_{j \geq 1} a_j x^j) = (A(x))^2.
\]

This means $A(x) - x = (A(x))^2$, and hence $A(x)^2 - A(x) + x = 0$. Solving the quadratic equation for $A(x)$, we obtain two possible solutions: $A(x) = (1 + \sqrt{1 - 4x})/2$ and $A(x) = (1 - \sqrt{1 - 4x})/2$. The former solution can be discarded, since it would give $a_0 = A(0) = 1$, which contradicts our assumption $a_0 = 0$. Thus,

\[
A(x) = \frac{1 - \sqrt{1 - 4x}}{2} = \frac{1}{2} - \frac{1}{2} (1 - 4x)^{1/2} = \frac{1}{2} - \frac{1}{2} \sum_{n \geq 0} \binom{1/2}{n} (-4x)^n.
\]

We infer from it $a_0 = 0$, and for $n \geq 1$,

\[
a_n = -\frac{1}{2} \binom{1/2}{n} (-4)^n.
\]

Note that, for $n \geq 2$

\[
\binom{1/2}{n} = \frac{1}{n!} \frac{1}{n} \left( \frac{1}{2} - 1 \right) \left( \frac{1}{2} - 3 \right) \cdots \left( \frac{1}{2} - (n - 1) \right)
= \frac{1}{n!} \frac{1}{2} \left( -\frac{3}{2} \right) \cdots \left( -\frac{2n - 3}{2} \right)
= \frac{1}{2^n n!} (1)^{n-1} \cdot 3 \cdots (2n - 3)
= \frac{1}{2^n n!} (1)^{n-1} \frac{1 \cdot 2 \cdot 3 \cdots (2n - 2)}{2 \cdot 4 \cdots (2n - 2)}
= \frac{1}{2^n n!} (1)^{n-1} \frac{(2n - 2)!}{2^{n-1}(n - 1)!}
= (-1)^{n-1} \frac{2}{4^n} \frac{1}{n!} (2n - 2)\binom{n}{n-1}.
\]

This leads to

\[
a_n = -\frac{1}{2} \binom{1/2}{n} (-4)^n
\]
$$= \frac{1}{2} (-4)^n \cdot (-1)^{n-1} \frac{1}{4^n} \binom{2n - 2}{n - 1}$$

$$= \frac{1}{n} \binom{2n - 2}{n - 1},$$

for \( n \geq 2 \). The above formula also holds for \( n = 1 \) since both sides are equal to 1. (Recall \( \binom{0}{0} = \frac{0!}{0!0!} = 1 \).) The numbers \( a_n \) are often called the Catalan numbers.