COS 341, October 18, 1999
Handout Number 3

Math Casino

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Let $N = 10^4$. Roulettes in the Math Casino have the following rule. For each run, a random integer n, $0 \le n < N$ is uniformly generated. If $\lfloor n^{1/4} \rfloor$ divides n, you win \$5; otherwise, the \$1 bet you place is forfeited. Should you play the game? One way to make a rational decision is to calculate the probability p of winning a game, and play the game if and only if $p \ge 1/6$. How can we compute p?

By definition $p = m/N$, where m is the number of integers $0 \le n < 10^4$ that satisfy "$\lfloor n^{1/4} \rfloor$ divides n". Let A_j be the set of integers n satisfying $\lfloor n^{1/4} \rfloor = j$ (i.e., $j^4 \le n \le (j+1)^4 - 1$) and "j dividing n". Clearly,

$$m = \sum_{0 \le j \le 9} |A_j|.$$

By inspection, we have $A_0 = \{0\}$ and hence $|A_0| = 1$. Let $j > 0$. Starting at $n = j^4$, every j-th integer in the range $j^4 \le n \le (j+1)^4 - 1$ satisfies j dividing n. Hence, for $j > 1$,

$$\begin{aligned} |A_j| &= \lceil \frac{(j+1)^4 - 1 - j^4 + 1}{j} \rceil \\ &= \lceil 4j^2 + 6j + 4 + \frac{1}{j} \rceil \\ &= 4j^2 + 6j + 4 + 1 \\ &= 4j^2 + 6j + 5. \end{aligned}$$

This shows

$$\begin{aligned} m &= 1 + \sum_{1 \le j \le 9} (4j^2 + 6j + 5) \\ &= 1 + 4 \sum_{1 \le j \le 9} j^2 + 6 \sum_{1 \le j \le 9} j + 9 \cdot 5 \\ &= 46 + 4 \sum_{1 \le j \le 9} j^2 + 6 \frac{1}{2} 9 \cdot 10 \\ &= 316 + 4 \sum_{1 \le j \le 9} j^2. \qquad (1) \end{aligned}$$

It remains to evaluate $a \equiv \sum_{1 \le j \le 9} j^2$. Using the equality $\sum_{2 \le k \le s} \binom{k}{2} = \binom{s+1}{3}$, we have

$$\begin{aligned} a &= \sum_{1 \le j \le 9} j^2 \\ &= \sum_{1 \le j \le 9} (2\binom{j}{2} + j) \\ &= 2 \sum_{1 \le j \le 9} \binom{j}{2} + \frac{1}{2} 9 \cdot 10 \\ &= 2\binom{9+1}{3} + 45 \\ &= 285. \end{aligned}$$

From (1) we then have

$$m = 316 + 4 \cdot 285 = 1456.$$

Hence $p = m/N = 0.1456 < 1/6$, and you shouldn't play this game. This solves the Math Casino Problem.