Name:
NetID:

COS 302 – Midterm Practice Questions

Problem 1 (???pts)
A family works together to knit scarves that will be donated to people in need. They have different skill levels: dad, mom, and their daughter Alice can do 40, 45, and 36 stitches per minute, respectively. Dad starts to knit 20 minutes later than mom and 100 minutes earlier than Alice. In the end, they finish at the same time and get six scarves in total. We know that there are 5600 stitches in each complete scarf. Assume everyone always has a constant knitting speed.

(A) The knitting task implies a linear system. Define variables for the total number of minutes spent by dad, mom and Alice, and write down the linear system that summarizes the conditions above using the variables defined.

(B) Solve the linear system using Gaussian Elimination.

(A) Let the total number of minutes spent by dad, mom and Alice be x, y, and z, respectively.

$$40x + 45y + 36z = 6 \times 5600$$
$$x - y = -20$$
$$x - z = 100$$

(B)

$$\left[\begin{array}{ccc|c} 40 & 45 & 36 & 33600 \\ 1 & -1 & 0 & -20 \\ 1 & 0 & -1 & 100 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & -20 \\ 40 & 45 & 36 & 33600 \\ 1 & 0 & -1 & 100 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & -20 \\ 0 & 85 & 36 & 34400 \\ 1 & 0 & -1 & 100 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & -20 \\ 0 & 85 & 36 & 34400 \\ 0 & 1 & -1 & 120 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -20 \\ 0 & 121 & 0 & 38720 \\ 0 & 1 & -1 & 120 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & -20 \\ 0 & 1 & 0 & 320 \\ 0 & 1 & -1 & 120 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & -20 \\ 0 & 1 & 0 & 320 \\ 0 & 1 & -1 & 120 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 300 \\ 0 & 1 & 0 & 320 \\ 0 & 1 & -1 & 120 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 300 \\ 0 & 1 & 0 & 320 \\ 0 & 0 & 1 & 200 \end{array}\right]$$

$x = 300$, $y = 320$, $z = 200$

Problem 2 (20pts)

Ryan's kids, Dorian and Eleanor, love sushi. Like all little kids, however, they really want to make sure things are fair and that neither of them gets to have more than the other one. Ryan orders six pieces of salmon sashimi and twelve Alaska rolls. Ellie and Dorian eat it all up, each eating an equal number of rolls. Can you figure out how many rolls of each type were eaten by each child? Let s_e and s_d be the number of salmon pieces eaten by Eleanor and Dorian, respectively, and let a_e and a_d be the respective number of Alaska rolls they ate.

(A) Write down a set of equations describing this problem.

(B) Is it possible to uniquely determine who ate what? Support your argument.

(C) Dorian complains *"Ellie got twice as many pieces of salmon as me!"*. Is the system now underdetermined, overdetermined, or uniquely solvable? Explain your answer.

(A)

$$\begin{aligned} s_e + s_d &= 6 && \text{all the salmon got eaten} \\ a_e + a_d &= 12 && \text{all of the Alaska rolls got eaten} \\ s_e + a_e &= 9 && \text{Ellie ate half} \\ s_d + a_d &= 9 && \text{Dorian ate the other half} \end{aligned}$$

(B) Not uniquely solvable. Reordering and putting in RREF:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & 12 \\ 1 & 1 & 0 & 0 & 9 \\ 0 & 0 & 1 & 1 & 9 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & 12 \\ 0 & 0 & 1 & 1 & 9 \\ 1 & 1 & 0 & 0 & 9 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & 12 \\ 0 & 0 & 1 & 1 & 9 \\ 0 & 1 & -1 & 0 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & 12 \\ 0 & 0 & 1 & 1 & 9 \\ 0 & 0 & -1 & -1 & -9 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & 12 \\ 0 & 0 & 1 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Reveals the rows are not linearly independent.

(C) The problem is now uniquely solvable. It adds equation $s_e - 2s_d = 0$. Replace the last equation with this one:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & 12 \\ 0 & 0 & 1 & 1 & 9 \\ 1 & -2 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & 12 \\ 0 & 0 & 1 & 1 & 9 \\ 0 & 0 & -3 & 0 & -6 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & 12 \\ 0 & 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right]$$

From this we can see that Dorian had 2 pieces of salmon, so Ellie had 4 pieces of salmon. Dorian thus had 7 Alaska rolls and Ellie had 5 Alaska rolls.

Problem 3 (??pts)

What is the range of values for a in the matrix below for which it is positive definite?

$$\begin{bmatrix} a & -1 \\ -1 & 4 \end{bmatrix}$$

Observe that the determinant of a positive definite matrix must be positive:

$$\begin{aligned} 4 \times a - (-1) \times (-1) &> 0 \\ 4a - 1 &> 0 \\ 4a &> 1 \\ a &> \frac{1}{4} \end{aligned}$$

Why can't both eigenvalues be negative in this case? If they were both negative, then the matrix would be negative definite, and so multiplying it by -1 would result in a positive definite matrix. However, we know that positive definite matrices must have positive diagonals, and so -4 would contradict this.

Problem 4 (??pts)

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be positive definite $n \times n$ matrices, and let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ where $\boldsymbol{x} \neq 0$ and $\boldsymbol{y} \neq 0$. Are the following statements true or false? Circle your answer.

(A) [TRUE] — FALSE $\boldsymbol{A} + \boldsymbol{B}$ is positive definite.

(B) [TRUE] — FALSE $\boldsymbol{x}^\mathsf{T} \boldsymbol{A} \boldsymbol{x} > 0$

(C) TRUE — [FALSE] $\det(\boldsymbol{A}) = 0$

(D) [TRUE] — FALSE $\boldsymbol{A}^\mathsf{T} \boldsymbol{A}$ is positive definite.

(E) TRUE — [FALSE] $\boldsymbol{A}$ is singular.

(F) TRUE — [FALSE]   The eigenvalues of $\boldsymbol{B}$ are negative.

Problem 5 (???pts)

Match the statements.

(A) A matrix that equals its transpose.

(B) A square symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ such that $\forall \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x} \geq 0$.

(C) A square matrix that has no inverse.

(D) For a linear map represented with matrix $\boldsymbol{A}$, the set of vectors $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$.

(E) For a linear map represented with matrix $\boldsymbol{A}$, the set of vectors $\boldsymbol{y}$ such that there exists an $\boldsymbol{x}$ where $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$.

(i) Positive semi-definite matrix
MATCH: (B)

(ii) Image
MATCH: (E)

(iii) Symmetric matrix
MATCH: (A)

(iv) Kernel/nullspace
MATCH: (D)

(v) Singular matrix
MATCH: (C)

Problem 6 (???pts)

Consider the following code snippet and match each variable with its value.

```
import numpy as np

def create_matrix(N):
    vec = np.arange(N) + 1
    return vec + vec.reshape(N, 1)

A = create_matrix(3)
B = A - np.mean(A, axis=0)
C = np.matmul(B, A[:, 2].reshape(3, 1))
D = np.sum(C)
```

(i)
```
0.0
```

variable name: `D`

(ii)
```
[[-15.]
 [  0.]
 [ 15.]]
```

variable name: `C`

(iii)
```
[[2 3 4]
 [3 4 5]
 [4 5 6]]
```

variable name: `A`

(iv)
```
[[-1. -1. -1.]
 [ 0.  0.  0.]
 [ 1.  1.  1.]]
```

variable name: `B`

Problem 7 (???pts) (A) Find the projection matrix $\mathbf{P}_\pi$ onto the line through the origin spanned by $\mathbf{b} = \begin{bmatrix}1\\2\\3\end{bmatrix}$.

(B) Find the projection $\mathbf{z}$ of the vector $\mathbf{x} = \begin{bmatrix}2\\2\\2\end{bmatrix}$ onto the subspace spanned by $\mathbf{b}$.

(C) Now use the result of (b) to find the projection of $\mathbf{z}$ onto the subspace spanned by $\mathbf{b}$.

(D) What is the rank of $\mathbf{P}_\pi$?

(E) Write down an eigenvector with corresponding eigenvalue of $\mathbf{P}_\pi$.

You may leave your answers in terms of unreduced fractions.

(A) $\mathbf{P}_\pi = \frac{1}{14}\begin{bmatrix}1 & 2 & 3\\2 & 4 & 6\\3 & 6 & 9\end{bmatrix}$

(B) $\mathbf{z} = \mathbf{P}_\pi \mathbf{x} = \frac{1}{14}\begin{bmatrix}12\\24\\36\end{bmatrix}$

(C) $\mathbf{P}_\pi^2 \mathbf{x} = \mathbf{z}$

(D) 1

(E) eigenvector: $\mathbf{z}$, eigenvalue: 1

Problem 8 (???pts)

Which of the following are valid bilinear maps for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$? Explain your reasoning.

(A) $f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^\top \boldsymbol{y} + 1$

(B) $f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^\top \boldsymbol{y}$

(C) $f(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{tr}(\boldsymbol{x}\boldsymbol{y}^\top \boldsymbol{A})$ for square symmetric $\boldsymbol{A}$

(D) $f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}\boldsymbol{y}^\top + \boldsymbol{y}\boldsymbol{x}^\top$

We require: $f(\lambda \boldsymbol{x} + \psi z, \boldsymbol{y}) = \lambda f(\boldsymbol{x}, \boldsymbol{y}) + \psi f(z, \boldsymbol{y})$ hold for both arguments.

(A) No. $\lambda \boldsymbol{x}^\top \boldsymbol{y} + 1 \neq \lambda(\boldsymbol{x}^\top \boldsymbol{y} + 1)$

(B) Yes. Simple inner product.

(C) Yes. Equal to $\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{x}$ due to cyclic property of trace.

(D) Yes:

$$\begin{aligned} f(\lambda \boldsymbol{x} + \psi \boldsymbol{y}, z) &= (\lambda \boldsymbol{x} + \psi \boldsymbol{y})z^\top + z(\lambda \boldsymbol{x} + \psi \boldsymbol{y})^\top \\ &= \lambda \boldsymbol{x}z^\top + \psi \boldsymbol{y}z^\top + \lambda z\boldsymbol{x}^\top + \psi z\boldsymbol{y}^\top \\ &= (\lambda \boldsymbol{x}z^\top + \lambda z\boldsymbol{x}^\top) + (\psi \boldsymbol{y}z^\top + \psi z\boldsymbol{y}^\top) \\ &= \lambda f(\boldsymbol{x}, z) + \psi f(\boldsymbol{y}, z) \end{aligned}$$

Problem 9 (???pts) (A) Rotate each of the following vectors by 45°:

$$\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(B) Check your work by computing the angle between $\boldsymbol{x}$ and its rotated version. Do the same with $\boldsymbol{y}$.

Rotation matrix is

$$\begin{bmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(A)

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(B) Normalized inner product produces cosine of angle.

$$\cos(\theta) = \frac{1}{\sqrt{2^2 + 3^2}} \frac{1}{\sqrt{(-1/\sqrt{2})^2 + (5/\sqrt{2})^2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{13}} \frac{1}{\sqrt{13}} \frac{1}{\sqrt{2}} 13 = \frac{1}{\sqrt{2}} = \cos(45°)$$

$$\cos(\theta) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} 1 = \frac{1}{\sqrt{2}} = \cos(45°)$$

Problem 10 (???pts)
Let

$$\boldsymbol{A} = \begin{bmatrix} 9 & 1 \\ 5 & 5 \end{bmatrix}.$$

(A) Compute the eigenvalues of $\boldsymbol{A}$.

(B) Find the eigenvectors of $\boldsymbol{A}$ for each of the eigenvalues you found in part (A).

(C) Let $\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Calculate $\boldsymbol{A}^n \boldsymbol{u}_0$, where $n \geq 0$ is an integer.

(A) Find zeros of the characteristic polynomial.

$$\begin{aligned} \det(\boldsymbol{A} - \lambda \mathbf{I}) &= 0 \\ (9-\lambda)(5-\lambda) - 5 &= 0 \\ 40 - 14\lambda + \lambda^2 &= 0 \\ (\lambda - 10)(\lambda - 4) &= 0 \\ &\implies \lambda_1 = 10,\ \lambda_2 = 4 \end{aligned}$$

(B) Find eigenvectors by solving the linear system: $(\boldsymbol{A} - \lambda \mathbf{I})\boldsymbol{v} = \mathbf{0}$ for each eigenvalue you in part (A). For $\lambda = 1$,

$$\begin{aligned} \begin{bmatrix} 9-10 & 1 \\ 5 & 5-10 \end{bmatrix} \boldsymbol{v}_1 &= \mathbf{0} \\ \begin{bmatrix} -1 & 1 \\ 5 & -5 \end{bmatrix} \boldsymbol{v}_1 &= \mathbf{0} \\ &\implies \\ \boldsymbol{v}_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

For $\lambda = 0.4$,

$$\begin{aligned} \begin{bmatrix} 9-4 & 1 \\ 5 & 5-4 \end{bmatrix} \boldsymbol{v}_2 &= \mathbf{0} \\ \begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix} \boldsymbol{v}_2 &= \mathbf{0} \\ &\implies \\ \boldsymbol{v}_2 &= \begin{bmatrix} -\frac{1}{5} \\ 1 \end{bmatrix}. \end{aligned}$$

(C) Perform eigen-decomposition on $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}$, where $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}$ from part (B) and $\boldsymbol{D}$ is a diagonal matrix with the two eigenvalues found in part (A).

$$
\begin{aligned}
\boldsymbol{P}^{-1} &= \begin{bmatrix} 1 & -\frac{1}{5} \\ 1 & 1 \end{bmatrix}^{-1} = \frac{5}{6}\begin{bmatrix} 1 & \frac{1}{5} \\ -1 & 1 \end{bmatrix} \\
\boldsymbol{A}^n &= \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1} \\
&= \frac{5}{6}\begin{bmatrix} 1 & -\frac{1}{5} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 10^n & 0 \\ 0 & 4^n \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{5} \\ -1 & 1 \end{bmatrix} \\
&= \frac{5}{6}\begin{bmatrix} 10^n & -\frac{4^n}{5} \\ 10^n & 4^n \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{5} \\ -1 & 1 \end{bmatrix} = \frac{5}{6}\begin{bmatrix} 10^n + \frac{4^n}{5} & \frac{10^n}{5} - \frac{4^n}{5} \\ 10^n - 4^n & \frac{10^n}{5} + 4^n \end{bmatrix}
\end{aligned}
$$

Problem 11 (??pt)

Let

$$\mathbf{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \mathbf{b} = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{c} = \begin{bmatrix}1\\-2\\1\end{bmatrix}, \mathbf{d} = \begin{bmatrix}-1\\0\\1\end{bmatrix}, U = \text{span}[\mathbf{a}, \mathbf{b}], V = \text{span}[\mathbf{a}, \mathbf{c}], \boldsymbol{A} = [\mathbf{a}, \mathbf{b}, \mathbf{c}]$$

Are the following statements true or false? Please circle TRUE or FALSE for each statement.

(A) TRUE — FALSE: $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are linearly independent.

(B) TRUE — FALSE: $\mathbf{a}, \mathbf{b}$ and $\mathbf{d}$ are linearly independent.

(C) TRUE — FALSE: U and $\mathbf{c}$ are orthogonal.

(D) TRUE — FALSE: V and $\mathbf{b}$ are orthogonal.

(E) TRUE — FALSE: The dimensions of both $\mathbf{a}$ and $\mathbf{c}$ are 3, so the dimension of V is 3.

(F) TRUE — FALSE: $\boldsymbol{A}$ is full rank.

(G) TRUE — FALSE: $\det(\boldsymbol{A}) = 0$

(H) TRUE — FALSE: If we perform elementary transformations on $\boldsymbol{A}$ until it's in Reduced Row-Echelon Form, then we will definitely obtain an identity matrix in the end.

(I) TRUE — FALSE: If we perform elementary transformations on $\boldsymbol{A}$ until it's in Row-Echelon Form, then for the values at the four corners of the derived matrix, there is only one value that we are not sure whether it is zero or non-zero.

(J) TRUE — FALSE: $\mathbf{d} \in U$

(K) TRUE — FALSE: $\boldsymbol{A} + \boldsymbol{A}^T$ is a square matrix as well as a symmetric matrix.

We have the following observations:

(A) T. $\lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c} = \mathbf{0}$ only has one trivial solution.

(B) F. $\mathbf{a} - 2\mathbf{b} = \mathbf{d}$

(C) T. $\mathbf{a}^T\mathbf{c} = \mathbf{b}^T\mathbf{c} = 0 \Rightarrow \mathbf{a} \perp \mathbf{c}, \mathbf{b} \perp \mathbf{c} \Rightarrow U = \text{span}[\mathbf{a}, \mathbf{b}] \perp \mathbf{c}$

(D) F. $\mathbf{a}^T\mathbf{b} \neq 0$

(E) F. The dimension of the vector space V equals the number of basis vectors, therefore should be 2.

(F) T. This can be derived from statement (A)

(G) F. This can be derived from statement (A)

(H) T. This can be derived from statement (F)

(I) T. Based on (F), we know that after the transformations, all the diagonal values should be non-zero and the bottom-left value should be zero, but the value at the top-right corner could be zero or non-zero.

(J) T. $\mathbf{a} - 2\mathbf{b} = \mathbf{d} \Rightarrow \mathbf{d} \in U = \text{span}[\mathbf{a}, \mathbf{b}]$

(K) T. By definition.

Problem 12 (???pts)
Let $\boldsymbol{V} = \{(a, b, c) \in \mathbb{R}^3$ such that $a = b^3\}$. Determine whether $\boldsymbol{V}$ is a subspace of $\mathbb{R}^3$.

In order to show that $\boldsymbol{V}$ is a subspace of $\mathbb{R}^3$, we need to show that $\mathbf{0} \in \boldsymbol{V}$, and that $\boldsymbol{V}$ is closed under addition and scalar multiplication. The $\mathbf{0}$ element is referred to as the neutral element by the book in section **2.4.1**.

We will show that $\boldsymbol{V}$ is not a subspace of $\mathbb{R}^3$ by showing that $\boldsymbol{V}$ is not closed under addition. Given $v_1 = (8, 2, 0)$, $v_2 = (27, 3, 0)$. It is easy to check that $v_1, v_2 \in \boldsymbol{V}$ since $2^3 = 8$ and $3^3 = 27$.$v_1 + v_2 = (35, 5, 0)$, but $5^3 \neq 35$. Therefore, $v_1 + v_2 \notin \boldsymbol{V}$. Therefore, $\boldsymbol{V}$ is not closed under addition and is not a subspace of $\mathbb{R}^3$.

Problem 13 (???pts)
Let $\boldsymbol{X} \in \mathbb{R}^{m \times n}$ have an SVD given by $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$. If $\boldsymbol{Q} \in \mathbb{R}^{m \times m}$ is an orthonormal matrix, what is the SVD of $\boldsymbol{Q}\boldsymbol{X} = \tilde{\boldsymbol{U}}\tilde{\boldsymbol{\Sigma}}\tilde{\boldsymbol{V}}^\top$?

$$\tilde{\boldsymbol{U}} = \boldsymbol{Q}\boldsymbol{U}$$
$$\tilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}$$
$$\tilde{\boldsymbol{V}} = \boldsymbol{V}$$

Is $\boldsymbol{Q}\boldsymbol{U}$ orthonormal?

$$(\boldsymbol{Q}\boldsymbol{U})(\boldsymbol{Q}\boldsymbol{U})^\top = \boldsymbol{Q}\boldsymbol{U}\boldsymbol{U}^\top\boldsymbol{Q}^\top = \boldsymbol{Q}\boldsymbol{Q}^\top = \mathbf{I}$$
$$(\boldsymbol{U}\boldsymbol{Q})(\boldsymbol{U}\boldsymbol{Q})^\top = \boldsymbol{U}\boldsymbol{Q}\boldsymbol{Q}^\top\boldsymbol{U}^\top = \boldsymbol{U}\boldsymbol{U}^\top = \mathbf{I}$$