COS 302 Precept 10

Spring 2020

Princeton University

Outline

Constrained Optimization

Lagrange Multipliers

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Unconstrained Optimization

Definition

For real-valued functions $f : R^D \rightarrow R$, an unconstrained optimization problem is:

$$\min_{x} \quad f(x) \tag{1}$$

Constrained Optimization

Definition

For real-valued functions $f, g_i : R^D \to R$ for $i = 1, 2, \cdots, m$, a constrained optimization problem is for all $i = 1, \cdots, m$:

$$\begin{aligned} \min_x \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \end{aligned} \tag{2}$$

This is known as the primal problem, corresponding to the primal variables x.

Rewriting the Primal Problem

Let

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i g_i(\boldsymbol{x}) \tag{3}$$

$$= f(\boldsymbol{x}) + \boldsymbol{\lambda}^T \boldsymbol{g}(\boldsymbol{x}) \tag{4}$$

Where $\lambda_i \geq 0 \ \forall i = 1, 2, \cdots, m$. Notice that the primal problem can be written as $\min_{\boldsymbol{x} \in R^d} \max_{\boldsymbol{\lambda} \geq 0} L(\boldsymbol{x}, \boldsymbol{\lambda})$

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Lagrange Duality

Definition

The associated Lagrangian dual problem with the primal problem is given by:

$$\begin{aligned} \max_{\boldsymbol{\lambda} \in R^m} \quad & D(\boldsymbol{\lambda}) \\ \text{subject to} \quad & \boldsymbol{\lambda} \geq 0 \end{aligned} \tag{5}$$

where $D(\boldsymbol{\lambda}) = \min_{x \in R^d} L(\boldsymbol{x}, \boldsymbol{\lambda})$, where we concatenated all constraints $g_i(\boldsymbol{x})$ into a single vector, $\boldsymbol{g}(\boldsymbol{x})$.

Minmax Ineqaulity

Definition

For any function with two arguments, $f(x, y)$, the maxmin is less than or equal to the minmax:

$$\max_{y} \min_{x} f(x, y) \leq \min_{x} \max_{y} f(x, y) \quad (6)$$

Minmax Ineqaulity Proof

The minmax ineqaulity can be proved by considering this inequality:

For all $\boldsymbol{x}, \boldsymbol{y}$, $\min_{\boldsymbol{x}} f(\boldsymbol{x}, \boldsymbol{y}) \leq f(\boldsymbol{x}, \boldsymbol{y}) \leq \max_{\boldsymbol{y}} f(\boldsymbol{x}, \boldsymbol{y})$

Weak Duality

The primal values are always greater than the Dual Value. That is,

$$\min_{\boldsymbol{x} \in R^d} \max_{\boldsymbol{\lambda} \geq 0} L(\boldsymbol{x}, \boldsymbol{\lambda}) \geq \max_{\boldsymbol{\lambda} \geq 0} \min_{\boldsymbol{x} \in R^d} L(\boldsymbol{x}, \boldsymbol{\lambda})$$

This is a direct application of the minmax inequality.