COS 302 Precept 2

Princeton University

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Definition of a Group

Definition 2.7 (Group)**.** Consider a set $\mathcal{G}$ and an operation $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ defined on $\mathcal{G}$. Then $G := (\mathcal{G}, \otimes)$ is called a *group* if the following hold:

1. *Closure* of $\mathcal{G}$ under $\otimes$: $\forall x, y \in \mathcal{G} : x \otimes y \in \mathcal{G}$
2. *Associativity:* $\forall x, y, z \in \mathcal{G} : (x \otimes y) \otimes z = x \otimes (y \otimes z)$
3. *Neutral element:* $\exists e \in \mathcal{G}\, \forall x \in \mathcal{G} : x \otimes e = x$ and $e \otimes x = x$
4. *Inverse element:* $\forall x \in \mathcal{G}\, \exists y \in \mathcal{G} : x \otimes y = e$ and $y \otimes x = e$, where e is the neutral element. We often write x^{-1} to denote the inverse element of x.

If additionally $\forall x, y \in \mathcal{G} : x \otimes y = y \otimes x$, then $G = (\mathcal{G}, \otimes)$ is an *Abelian group* (commutative).

Groups and Vector Spaces

Example: Vectors in $\mathbb{R}^n$ under addition

1. **Closure**: $\overrightarrow{a}, \overrightarrow{b} \in \mathbb{R}^n \Rightarrow \overrightarrow{a} + \overrightarrow{b} \in \mathbb{R}^n$
2. **Associativity**: $\overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c}) = (\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c}$
3. **Neutral element**: $\overrightarrow{a} + \overrightarrow{0} = \overrightarrow{a}$
4. **Inverse element:** $\overrightarrow{a} + -\overrightarrow{a} = 0$
5. (abelian) **Commutativity:** $\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{b} + \overrightarrow{a}$

Groups and Vector Spaces

Definition 2.9 (Vector Space). A real-valued *vector space* $V = (\mathcal{V}, +, \cdot)$ is a set $\mathcal{V}$ with two operations

$$\text{“Inner Operation”} \rightarrow + : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$$

$$\text{“Outer Operation” “Scaling”} \rightarrow \cdot : \mathbb{R} \times \mathcal{V} \rightarrow \mathcal{V}$$

where

1. $(\mathcal{V}, +)$ is an Abelian group
2. Distributivity:
   1. $\forall \lambda \in \mathbb{R}, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V} : \lambda \cdot (\boldsymbol{x} + \boldsymbol{y}) = \lambda \cdot \boldsymbol{x} + \lambda \cdot \boldsymbol{y}$
   2. $\forall \lambda, \psi \in \mathbb{R}, \boldsymbol{x} \in \mathcal{V} : (\lambda + \psi) \cdot \boldsymbol{x} = \lambda \cdot \boldsymbol{x} + \psi \cdot \boldsymbol{x}$
3. Associativity (outer operation): $\forall \lambda, \psi \in \mathbb{R}, \boldsymbol{x} \in \mathcal{V} : \lambda \cdot (\psi \cdot \boldsymbol{x}) = (\lambda\psi) \cdot \boldsymbol{x}$
4. Neutral element with respect to the outer operation: $\forall \boldsymbol{x} \in \mathcal{V} : 1 \cdot \boldsymbol{x} = \boldsymbol{x}$

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Row-Echelon Form

Definition

A matrix is in row-echelon form if:

- All rows that contain only zeros are at the bottom of the matrix.[a]
- Looking at nonzero rows only, the pivot[b] is always strictly to the right of the pivot of the row above it.

[a] Correspondingly, all rows that contain at least one nonzero element are on top of rows that contain only zeros.

[b] the first nonzero value from the left, also called the leading coefficient.

Row-Echelon Form

Examples

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Reduced Row-Echelon Form

Definition

A matrix is in reduced row-echelon form if

- It is in row-echelon form
- Every pivot[a] is 1
- The pivot is the only nonzero entry in its column.

[a] The first nonzero value from the left in each row

Reduced Row-Echelon Form

Examples

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & 3 & 0 & 0 & 3\\0 & 0 & 1 & 0 & 9\\0 & 0 & 0 & 1 & -4\end{bmatrix} \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\end{bmatrix} \begin{bmatrix}1 & 0 & 8 & -4\\0 & 1 & 2 & 12\end{bmatrix}$$

In general, row-echelon form and reduced row-echelon form make it easier for us to determine a particular solution and the general solution.

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Elementary Transformations

Given a matrix $\boldsymbol{A}$, there are three elementary operations one can perform on $\boldsymbol{A}$ to transform $\boldsymbol{A}$ into reduced row-echelon form without changing the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$.

- Addition of two rows

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- Exchange two rows of a matrix
- Exchange two columns of a matrix

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Gaussian Elimination

Gaussian elimination is an algorithm that performs elementary transformations to bring a system of linear equations into reduced row-echelon form.

Gaussian Elimination

$$\begin{cases} x_1 + x_2 - x_3 = 7 \\ x_1 - x_2 + 2x_3 = 3 \\ 2x_1 + x_2 + x_3 = 9 \end{cases}$$

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The above system of equations can be represented by this augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 1 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{array}\right]$$

We will perform Gaussian Elimination on this system of equations (Open Colab Notebook)

Invert Matrix via Gaussian Elimination

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Invert Matrix via Gaussian Elimination

Perform Gaussian Elimination on the following Augmented Matrix:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Invert Matrix via Gaussian Elimination

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 2 \end{array}\right]$$

Invert Matrix via Gaussian Elimination

$$A^{-1} = \begin{bmatrix} -1 & 2 & -2 & 2 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Justification(Optional)

Each elementary operation on $\boldsymbol{A}$ can be written as left multiplying $\boldsymbol{A}$ by a matrix. Transforming $\boldsymbol{A}$ to the identity matrix can be written as: $\boldsymbol{E_1 E_2 \cdots E_n A = I}$. This implies that $\boldsymbol{E_1 E_2 \cdots E_n A A^{-1} = I A^{-1} = A^{-1}}$, which implies that $\boldsymbol{E_1 E_2 \cdots E_n I = A^{-1}}$. This means that applying the sequence of elementary operations that transformed $\boldsymbol{A}$ to the identity matrix on $\boldsymbol{I}$ will transform $\boldsymbol{I}$ to $\boldsymbol{A^{-1}}$.