Common Probabilistic Identities and Inequalities

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Inequalities

Markov's inequality: Nonnegative random variable X

$$\text{For any } a > 0: \qquad P(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

Chebyshev's inequality: Random variable X with finite mean and variance

$$\text{For all } a > 0: \qquad P(|X - \mathbb{E}[X]| \geq a) \leq \frac{\mathrm{Var}(X)}{a^2}$$

Chernoff bound: Random variable X with mean zero

$$P(X \geq \epsilon) \leq \min_{t>0} \mathbb{E}[e^{tX}]\, e^{-t\epsilon} = \min_{t>0} M_X(t)\, e^{-t\epsilon}$$

Jensen's inequality: Random variable X, convex function $f(x)$

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$$

Central Limit Theorem

Independent, identically-distributed (i.i.d.) random variables $X_1, X_2, \ldots, X_n$ with mean μ and variance σ^2

$$\sqrt{n}\left(\frac{1}{n}\sum_i (X_i - \mu)\right) \text{ converges in distribution to } \mathcal{N}(0, \sigma^2) \text{ as } n \to \infty$$

or

$$\frac{1}{n}\sum_i X_i \text{ converges in distribution to } \mathcal{N}(\mu, \sigma^2/n) \text{ as } n \to \infty$$

- Handy rule: variance of the mean of i.i.d. random variables goes as $1/n$; standard deviation decreases as $1/\sqrt{n}$

Moment Generating Functions

Moment generating function for random variable X

$$M_X(t) = \mathbb{E}[e^{tX}] = \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{t^k X^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbb{E}[X^k]$$

- k^{th} moment is k^{th} derivative of $M_X(t)$ evaluated at $t = 0$:

$$\mathbb{E}[X^k] = \frac{d^k}{dt^k} M_X(0)$$

- MGF interacts nicely with scaling/sum:

$$Z = \alpha X + \beta Y$$

$$M_Z(t) = M_X(\alpha t)\, M_Y(\beta t)$$