Matrix Trace and Invariants

Szymon Rusinkiewicz

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PRINCETON UNIVERSITY

That Mysterious Trace…

- Simple definition: *trace* of a square matrix = sum of its diagonal elements
- Book properties:
  - Linearity: $\mathrm{Tr}(\boldsymbol{A}+\boldsymbol{B}) = \mathrm{Tr}(\boldsymbol{A}) + \mathrm{Tr}(\boldsymbol{B})$; $\quad \mathrm{Tr}(\alpha\boldsymbol{A}) = \alpha\,\mathrm{Tr}(\boldsymbol{A})$
  - Commutativity: $\mathrm{Tr}(\boldsymbol{AB}) = \mathrm{Tr}(\boldsymbol{BA})$ $\quad$ but $\neq \mathrm{Tr}(\boldsymbol{A})\,\mathrm{Tr}(\boldsymbol{B})$
- Other trivial properties:
  - For an n-dimensional identity matrix: $\mathrm{Tr}(\boldsymbol{I}) = n$
  - For a transpose: $\mathrm{Tr}(\boldsymbol{A}^\top) = \mathrm{Tr}(\boldsymbol{A})$

- But what's the intuition?

Invariance

- For any change-of-basis matrix $\boldsymbol{M}$,

$$\text{Tr}(\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}) = \text{Tr}(\boldsymbol{A})$$

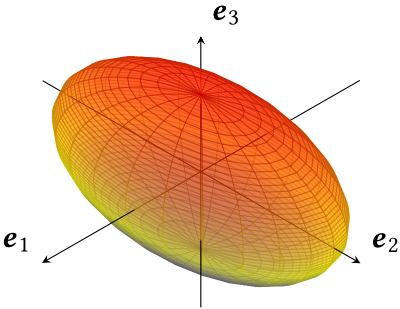
- Really big deal: this means that, like determinant, trace is *basis invariant*
- In particular, $\boldsymbol{M}$ can be transformation that diagonalizes into eigenbasis:

$$\boldsymbol{D} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$$

- So, trace equals the *sum of eigenvalues*, just as determinant is their product

Applications

- Consider a symmetric matrix A. You may recall that it always has real eigenvalues and orthogonal eigenvectors.
- If positive definite: think of generalized ellipse / ellipsoid



Applications

- Now consider the quadratic form

$$\hat{\boldsymbol{v}}^{\mathsf{T}} A\, \hat{\boldsymbol{v}}$$

  that tells you stretch along each unit-length direction $\hat{\boldsymbol{v}}$.

- $\frac{1}{n}$ times the trace of A gives the *mean* or *expected value* of the quadratic form over all directions $\hat{\boldsymbol{v}}$
  - In engineering, if matrix is *stress tensor*, gives mean stress
  - In differential geometry, if matrix is *curvature tensor*, gives mean curvature

Other Invariants

- You might wonder if there are other invariant quantities for a matrix

$$f(M^{-1}AM) = f(A)$$

  besides the trace and determinant.

- It turns out that for an $n \times n$ matrix there are n independent invariants (in the sense that they are not related to each other by some function).
  - For 2×2, just the trace and determinant!

Other Invariants

The *principal invariants* of a matrix $\boldsymbol{A}$ are:

For 2×2 :	For 3×3 :	For 4×4 :
$I_1 = \lambda_1 + \lambda_2$ $I_2 = \lambda_1\lambda_2$	$I_1 = \lambda_1 + \lambda_2 + \lambda_3$ $I_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ $I_3 = \lambda_1\lambda_2\lambda_3$	$I_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ $I_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4$ $I_3 = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4$ $I_4 = \lambda_1\lambda_2\lambda_3\lambda_4$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the eigenvalues of $\boldsymbol{A}$.

Notice that the first and last ones are always the trace and determinant.