Linear Mappings and $*$-ective $*$-morphisms

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a.k.a. The COS 302 Mathematician-English Dictionary

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Linear Mapping

The Mathematician Says:

A *vector space homomorphism* $\Phi : \mathbb{V} \longrightarrow \mathbb{W}$ satisfies

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}, \lambda, \psi \in \mathbb{R} : \Phi(\lambda \boldsymbol{x} + \psi \boldsymbol{y}) = \lambda \Phi(\boldsymbol{x}) + \psi \Phi(\boldsymbol{y})$$

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In English:

Φ is a *linear mapping* from $\mathbb{V}$ to $\mathbb{W}$ if it preserves the properties of a vector space.

If vectors in $\mathbb{V}$ are n-dimensional, and vectors in $\mathbb{W}$ are m-dimensional, then Φ can be represented by an $m \times n$ matrix.

Injectivity

The Mathematician Says:

An *injective* mapping $\Phi : \mathbb{V} \longrightarrow \mathbb{W}$ satisfies

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{V} : \Phi(\boldsymbol{x}) = \Phi(\boldsymbol{y}) \implies \boldsymbol{x} = \boldsymbol{y}$$

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In English:

Φ is *one-to-one* iff it doesn't collapse multiple elements into one.

Surjectivity

The Mathematician Says:

An *surjective* mapping $\Phi : \mathbb{V} \to \mathbb{W}$ satisfies

$$\forall \boldsymbol{w} \in \mathbb{W}, \exists \boldsymbol{v} \in \mathbb{V} : \Phi(\boldsymbol{v}) = \boldsymbol{w}$$

Surjectivity

The Mathematician Says:

An *surjective* mapping $\Phi : \mathbb{V} \longrightarrow \mathbb{W}$ satisfies

$$\forall \boldsymbol{w} \in \mathbb{W}, \exists \boldsymbol{v} \in \mathbb{V} : \Phi(\boldsymbol{v}) = \boldsymbol{w}$$

In English:

Φ is *onto* iff it can output every element of $\mathbb{W}$ (perhaps not uniquely).

Bijectivity

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A mapping $\Phi : \mathbb{V} \longrightarrow \mathbb{W}$ is *bijective* iff it is both injective and surjective.

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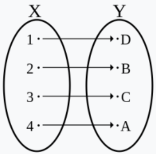
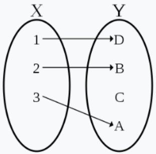
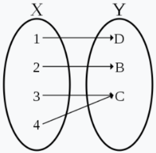
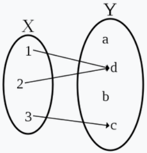
In English:

Φ is a *one-to-one correspondence* iff there is
a unique element of $\mathbb{W}$ for every element of $\mathbb{V}$, and vice versa.

In this case, Φ is guaranteed to have an *inverse*, written Φ^{-1}.

∗-jectivity Decoder Chart

	surjective	non-surjective
injective	 **bijective**	 **injective-only**
non-injective	 **surjective-only**	 **general**

From `https://en.wikipedia.org/wiki/Bijection,_injection_and_surjection`

$*$-morphisms

- Isomorphisms are linear, bijective maps.
- Endomorphisms are linear maps into the same space. (“square matrix”)
- Automorphisms are isomorphic endomorphisms. (“square invertible matrix”)

Kernel

The Mathematician Says:

The *kernel* of a mapping $\Phi : \mathbb{V} \longrightarrow \mathbb{W}$ satisfies

$$\ker(\Phi) = \{\boldsymbol{v} \in \mathbb{V} : \Phi(\boldsymbol{v}) = \mathbf{0}\}$$

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In English:

A linear mapping collapses some set of vectors (always including the $\mathbf{0}$ vector) to zero. This set of vectors is called the *kernel* or *null space*.

Image

The Mathematician Says:

The *image* of a mapping $\Phi : \mathbb{V} \longrightarrow \mathbb{W}$ satisfies

$$\mathrm{Im}(\Phi) = \{\boldsymbol{w} \in \mathbb{W} : \exists \boldsymbol{v} \in \mathbb{V}, \Phi(\boldsymbol{v}) = \boldsymbol{w}\}$$

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In English:

The *image* or *range* of a mapping is the set of vectors it can output.

Rank-Nullity Theorem

The Mathematician Says:

The dimension of the domain of a linear map is the sum of the dimensions of its kernel and its image.

$$\dim(\mathbb{V}) = \dim\big(\ker(\Phi)\big) + \dim\big(\mathrm{Im}(\Phi)\big)$$

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$$\dim(\mathbb{V}) = \dim\big(\ker(\Phi)\big) + \dim\big(\mathrm{Im}(\Phi)\big)$$

In English:

The number of dimensions preserved by a linear transformation, plus the number collapsed to zero, equals the dimension of the original vector space (where the "dimensions" need not be the coordinate axes or basis vectors.)