Linear Independence, Bases, and Rank

Szymon Rusinkiewicz

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PRINCETON UNIVERSITY

Linear Combination

- Let $\mathbb{V}$ be a vector space. $\boldsymbol{v} \in \mathbb{V}$ is a *linear combination* of vectors $\boldsymbol{x_1}, \ldots, \boldsymbol{x_k} \in \mathbb{V}$ if

$$\boldsymbol{v} = \lambda_1 \boldsymbol{x_1} + \ldots + \lambda_k \boldsymbol{x_k} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x_i} \in \mathbb{V}$$

- *Nontrivial* linear combinations have at least one coefficient $\lambda_i \neq 0$
  - The $\mathbf{0}$-vector can be "trivially" represented as a linear combination $\sum_{i=1}^{k} 0\boldsymbol{x_i}$.

Linear (In)dependence

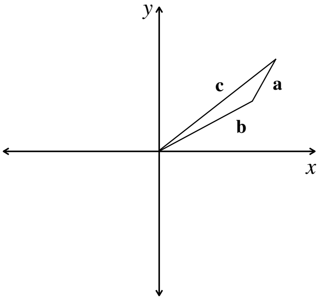
- If there is at least one nontrivial linear combination of $\boldsymbol{x_1}, \ldots, \boldsymbol{x_k} \in \mathbb{V}$ such that $\sum_{i=1}^{k} \lambda_i \boldsymbol{x_i} = \mathbf{0}$, then $\boldsymbol{x_1}, \ldots, \boldsymbol{x_k}$ are *linearly dependent*.
- Otherwise, when only the trivial solution exists, they are *linearly independent*.

Linear (In)dependence

- Linearly independent vectors have no “redundancy”
  - If we remove any one of them, there will be certain vectors we can no longer represent via linear combinations.
- Equivalently, can’t express any $\boldsymbol{x}_i$ as a linear combination of the others.

Linear (In)dependence

Example: Consider three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ where $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$.



These vectors are linearly dependent because $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c} = \boldsymbol{0}$.

Checking Linear Independence

Use Gaussian Elimination to check linear (in-)dependence:

- Construct a matrix by stacking the vectors as columns
- Reduce to row echelon form
- If every column has a leading "1," linearly independent

Checking Linear Independence

Example:

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}.$$

Transform the corresponding matrix to reduced row echelon form:

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ -3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Every column has a leading "1," so the vectors are linearly independent.

Span and Basis

- The span of a set of vectors is the set of all their linear combinations.
- A set of vectors is a generating set for a vector space $\mathbb{V}$ if its span is $\mathbb{V}$.
- A basis is a minimal generating set.

Basis Example

In $\mathbb{R}^3$, the *canonical* or *standard* basis is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Two other bases of $\mathbb{R}^3$ are

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \mathcal{B}_2 = \left\{ \begin{bmatrix} 0.5 \\ 0.8 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 1.8 \\ 0.3 \\ -0.3 \end{bmatrix}, \begin{bmatrix} -2.2 \\ -1.3 \\ 3.5 \end{bmatrix} \right\}$$

Basis Non-Example

The set

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \end{bmatrix} \right\}$$

is not a generating set (and so not a basis) of $\mathbb{R}^4$.

Remarks about Bases

- Every vector space possesses a basis, but there is no unique basis.
- All bases contain the same number of *basis vectors*.
- The *dimension* of $\mathbb{V}$ is the number of basis vectors of $\mathbb{V}$: intuitively, the dimension of a vector space can be thought of as the number of independent directions in this vector space.
- The dimension of a vector space is not *necessarily* the number of elements in a vector. For example,

$$\mathbb{V} = \text{span}\left(\begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix}\right)$$

  is one-dimensional.

Finding a Basis

Use Gaussian Elimination to find a basis for the vector space spanned by $\boldsymbol{x}_1 \ldots \boldsymbol{x}_m$:

- Construct a matrix by stacking the vectors as columns
- Reduce to row echelon form
- Take every column with a leading “1”

Finding Basis: Example

Consider a subspace $\mathbb{U}$ of $\mathbb{R}^5$ spanned by the vectors

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \quad \boldsymbol{x}_4 = \begin{bmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{bmatrix}$$

To find which of the vectors are a basis for $\mathbb{U}$...

Finding Basis: Example

Write down matrix and reduce:

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns (with leading ones) indicate linearly independent vectors, so $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_4$ form a basis for $\mathbb{U}$.

Rank

- The rank of a matrix is the number of linearly independently rows (= the number of linearly independent columns)
- **Example:** The matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

has $\mathrm{rk}(A) = 2$, because A has two linearly independent columns / rows.

Properties of Rank

- $\text{rk}(\boldsymbol{A}) = \text{rk}(\boldsymbol{A}^\top)$
- The columns of $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ span a subspace $\mathbb{U} \subseteq \mathbb{R}^m$ with $\dim(\mathbb{U}) = \text{rk}(\boldsymbol{A})$
- The rows of $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ span a subspace $\mathbb{W} \subseteq \mathbb{R}^n$ with $\dim(\mathbb{W}) = \text{rk}(\boldsymbol{A})$.
- For all $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{A}$ is invertible iff $\text{rk}(\boldsymbol{A}) = n$

Properties of Rank, cont.

- For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, the subspace of solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has dimension $n - \text{rk}(\boldsymbol{A})$. This is called the *kernel* or *null* space of $\boldsymbol{A}$.
- A matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ has *full rank* if its rank equals the largest possible rank for a matrix of the same dimensions. In other words, the rank of a full rank matrix is $\text{rk}(\boldsymbol{A}) = \min(m, n)$.
- A matrix is said to be *rank deficient* if it does not have full rank.
- A square matrix is *singular* if it does not have an inverse or, equivalently, is rank deficient.