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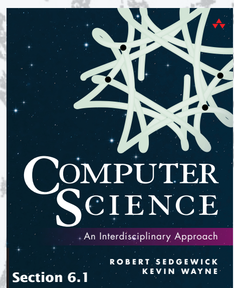

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19. Combinational Circuits

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19. Combinational Circuits

CS.19.A.Circuits.Basics

Context

Q. What is a combinational circuit?

A. A digital circuit (all signals are 0 or 1) with no feedback (no loops).

analog circuit: signals vary continuously

sequential circuit: loops allowed (stay tuned)

Q. Why combinational circuits?

A. Accurate, reliable, general purpose, fast, cheap.

Basic abstractions

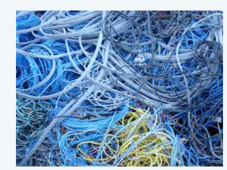
- On and off.
- Wire: propagates on/off value.
- Switch: controls propagation of on/off values through wires.

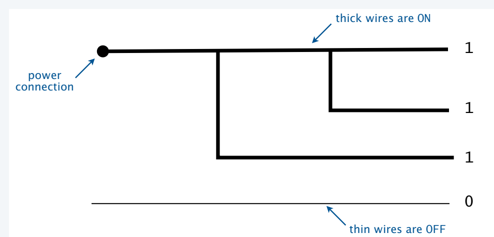
Applications. Smartphone, tablet, game controller, antilock brakes, *microprocessor*, ...

Wires

Wires propagate on/off values

- ON (1): connected to power.
- OFF (0): not connected to power.
- Any wire connected to a wire that is ON is also ON.
- Drawing convention: "flow" from top, left to bottom, right.

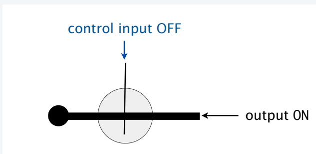


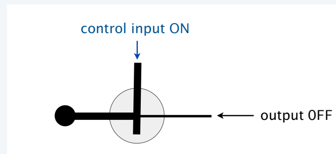


Controlled Switch

Switches control propagation of on/off values through wires.

- Simplest case involves two connections: control (input) and output.
- control OFF: output ON
- control ON: output OFF

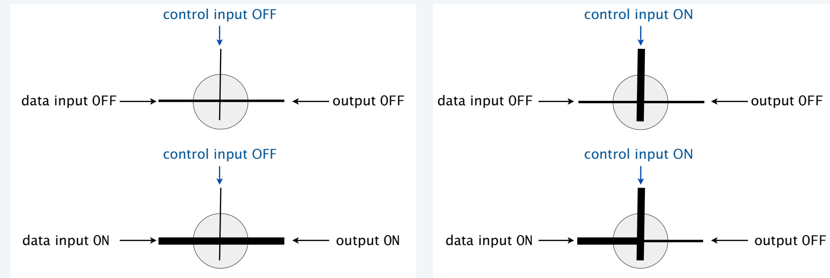




Controlled Switch

Switches control propagation of on/off values through wires.

- General case involves *three* connections: control input, *data input* and output.
- control OFF: output is connected to input
- control ON: output is disconnected from input

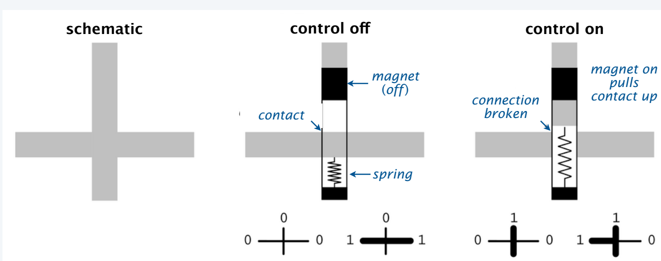


Idealized model of *pass transistors* found in real integrated circuits.

Controlled switch: example implementation

A *relay* is a physical device that controls a switch with a magnet

- 3 connections: input, output, control.
- Magnetic force pulls on a contact that cuts electrical flow.



First level of abstraction

Switches and wires model provides separation between physical world and logical world.

- We assume that switches operate as specified.
- That is the only assumption.
- Physical realization of switch is irrelevant to design.

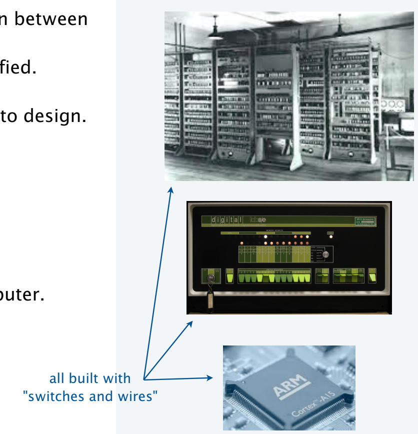
Physical realization dictates *performance*

- Size.
- Speed.
- Power.

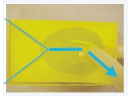
New technology immediately gives new computer.

Better switch? Better computer.

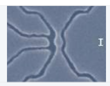
Basis of Moore's law.



Switches and wires: a first level of abstraction

technology	*"information"*	*switch*
pneumatic	air pressure	
fluid	water pressure	
relay (now)	electric potential	

Amusing attempts that do not scale but prove the point

technology	*switch*
relay (1940s)	
vacuum tube	
transistor	
"pass transistor" in integrated circuit	
atom-thick transistor	

Real-world examples that prove the point

Switches and wires: a first level of abstraction

VLSI = Very Large Scale Integration

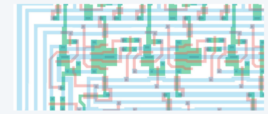
Technology
Deposit materials on substrate.

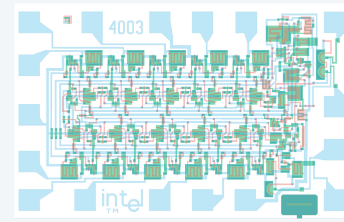
Key properties
Lines are wires.
Certain crossing lines are controlled switches.

Key challenge in physical world
Fabricating physical circuits with billions of wires and controlled switches

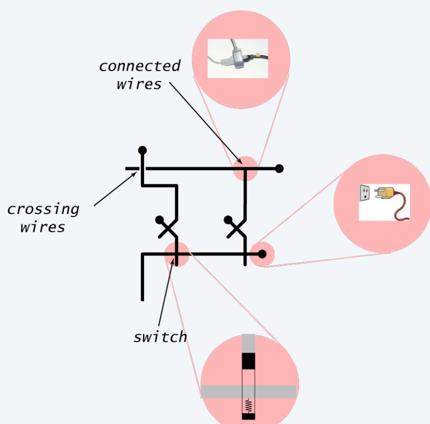
Key challenge in "abstract" world
Understanding behavior of circuits with billions of wires and controlled switches

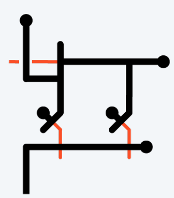
Bottom line. Circuit = Drawing (!)





Circuit anatomy





Need more levels of abstraction to understand circuit behavior

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Image sources

http://upload.wikimedia.org/wikipedia/commons/f/f4/1965_c1960s_vacuum_tube%2C_7025A-12AX7A%2C_QC%2C_Philips%2C_Great_Britain.jpg
http://electronics.howstuffworks.com/relay.htm

CS.19.A.Circuits.Basics

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19. Combinational Circuits

- Building blocks
- **Boolean algebra**
- Digital circuits
- Adder circuit
- Arithmetic/logic unit

CS.19.B.Circuits.Algebra

Boolean algebra

Developed by George Boole in 1840s to study logic problems
- Variables represent *true* or *false* (1 or 0 for short).
- Basic operations are AND, OR, and NOT (see table below).

Widely used in mathematics, logic and computer science.


George Boole
1815–1864

operation	Java notation	logic notation	circuit design (this lecture)
AND	x && y	$x \wedge y$	xy
OR	x || y	$x \vee y$	$x + y$
NOT	! x	$\neg x$	x'

various notations in common use

Example: (stay tuned for proof)

DeMorgan's Laws

$(xy)' = (x' + y')$
$(x + y)' = x'y'$

Relevance to circuits. Basis for next level of abstraction.

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Truth tables

A truth table is a systematic way to define a Boolean function
- One row for each possible set of arguments.
- Each row gives the function value for the specified arguments.
- N inputs: 2^N rows needed.

NOT

x	x'
0	1
1	0

AND

x	y	xy
0	0	0
0	1	0
1	0	0
1	1	1

OR

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

NOR

x	y	NOR
0	0	1
0	1	0
1	0	0
1	1	0

XOR

x	y	XOR
0	0	0
0	1	1
1	0	1
1	1	0

Truth table proofs

Truth tables are convenient for establishing identities in Boolean logic
- One row for each possibility.
- Identity established if columns match.

Proofs of DeMorgan's laws

x	y	xy	$(xy)'$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

x	y	x'	y'	$x' + y'$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

$(xy)' = (x' + y')$

x	y	$x + y$	NOR $(x + y)'$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

x	y	x'	y'	NOR $x'y'$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

$(x + y)' = x'y'$

All Boolean functions of two variables

Q. How many Boolean functions of two variables?

A. 16 (all possibilities for the 4 bits in the truth table column).

Truth tables for all Boolean functions of 2 variables

x	*y*	*ZERO*	*AND*		*x*		*y*	*XOR*	*OR*	*NOR*	*EQ*	*¬y*		*¬x*		*NAND*	*ONE*
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Functions of three and more variables

Q. How many Boolean functions of *three* variables?

A. 256 (all possibilities for the 8 bits in the truth table column).

x	*y*	*z*	*AND*	*OR*	*NOR*	*MAJ*	*ODD*
0	0	0	0	0	1	0	0
0	0	1	0	1	0	0	1
0	1	0	0	1	0	0	1
0	1	1	0	1	0	1	0
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	0
1	1	0	0	1	0	1	0
1	1	1	1	1	0	1	1

Some Boolean functions of 3 variables

Examples

all extend to *N* variables

AND	logical AND	0 iff *any* inputs is 0 (1 iff all inputs 1)
OR	logical OR	1 iff *any* input is 1 (0 iff all inputs 0)
NOR	logical NOR	0 iff *any* input is 1 (1 iff all inputs 0)
MAJ	majority	1 iff more inputs are 1 than 0
ODD	odd parity	1 iff an odd number of inputs are 1

Q. How many Boolean functions of *N* variables?

A. $2^{(2^N)}$

N	*number of Boolean functions with N variables*
2	$2^4 = 16$
3	$2^8 = 256$
4	$2^{16} = 65,536$
5	$2^{32} = 4,294,967,296$
6	$2^{64} = 18,446,744,073,709,551,616$

Universality of AND, OR and NOT

Every Boolean function can be represented as a sum of products
- Form an AND term for each 1 in Boolean function.
- OR all the terms together.

x'yz + xy'z + xyz' + xyz = MAJ

x	*y*	*z*	*MAJ*	*x'yz*	*xy'z*	*xyz'*	*xyz*	
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	1
1	0	0	0	0	0	0	0	0
1	0	1	1	0	1	0	0	1
1	1	0	1	0	0	1	0	1
1	1	1	1	0	0	0	1	1

Expressing MAJ as a sum of products

Def. A set of operations is *universal* if every Boolean function can be expressed using just those operations.

Fact. { AND, OR, NOT } is universal.

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Image sources

http://en.wikipedia.org/wiki/George_Boole#/media/File:George_Boole_color.jpg

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19. Combinational Circuits

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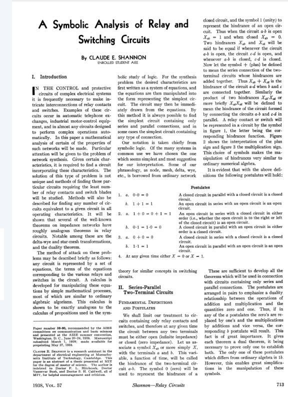
- Boolean algebra
- **Digital circuits**
- Adder circuit
- Arithmetic/logic unit

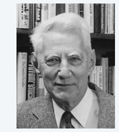
CS.19.C.Circuits.Digital

A basis for digital devices

Claude Shannon connected *circuit design* with Boolean algebra in 1937.

"Possibly the most important, and also the most famous, master's thesis of the [20th]
– Howard Gardner




Claude Shannon
1916–2001

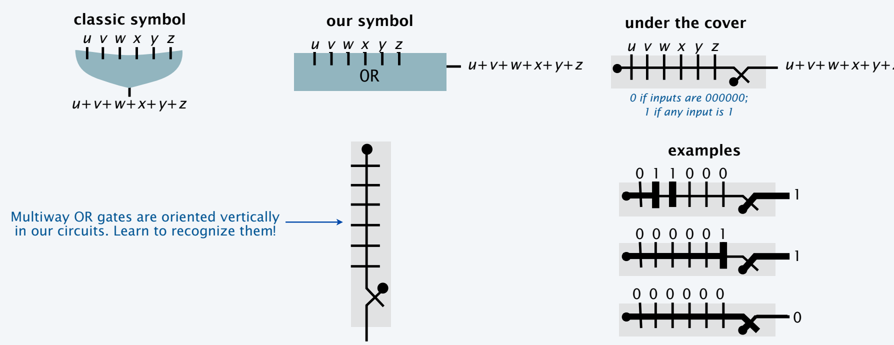
Key idea. Can use Boolean algebra to systematically analyze circuit behavior.

A second level of abstraction: logic gates

boolean function	notation	truth table	classic symbol	our symbol	under the cover circuit (gate)	proof
NOT	x'	x x' / 0 1 / 1 0	x → x'	x — ¬ — x'	x x'	*1 iff x is 0*
NOR	(x + y)'	x y NOR / 0 0 1 / 0 1 0 / 1 0 0 / 1 1 0	x y (x+y)'	x y NOR (x+y)'	x y (x+y)'	*1 iff x and y are both 0*
OR	x + y	x y OR / 0 0 0 / 0 1 1 / 1 0 1 / 1 1 1	x y x+y	x y OR x+y	x y x+y	NOR ¬ x+y = ((x + y)')'
AND	xy	x y AND / 0 0 0 / 0 1 0 / 1 0 0 / 1 1 1	x y xy	x y AND xy	x y xy	¬ ¬ NOR xy = (x' + y')'

Multiway OR gates

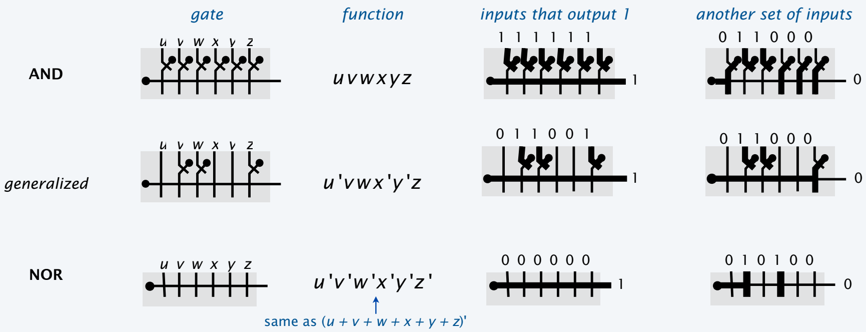
OR gates with multiple inputs.
- 1 if *any* input is 1.
- 0 if *all* inputs are 0.



Multiway generalized AND gates

Multiway generalized AND gates.

- 1 for *exactly 1* set of input values.
- 0 for *all other* sets of input values.

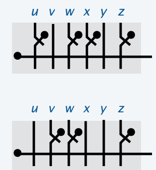


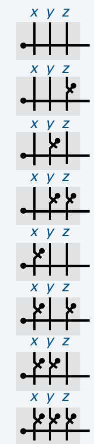
Might also call these "generalized NOR gates"; we consistently use AND.

Pop quiz on generalized AND gates

Q. Give the Boolean function computed by these gates.

Q. Also give the inputs for which the output is 1.

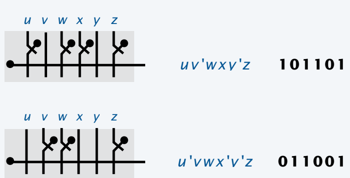


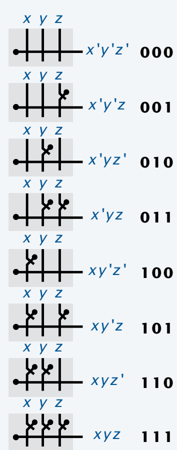


Pop quiz on generalized AND gates

Q. Give the Boolean function computed by these gates.

Q. Also give the inputs for which the output is 1.





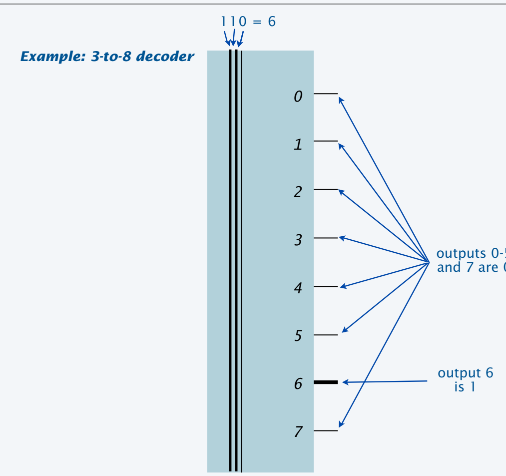
Get the idea? If not, replay this slide, like flash cards.

Note. From now on, we will not label these gates.

A useful combinational circuit: decoder

Decoder

- n input lines (address).
- 2^n outputs.
- Addressed output is 1.
- All other outputs are 0.



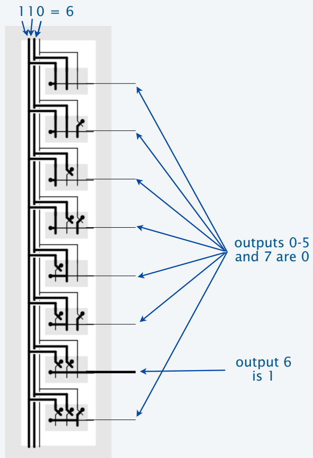
A useful combinational circuit: decoder

Decoder
- *n* input lines (address).
- 2^n outputs.
- Addressed output is 1.
- All other outputs are 0.

Implementation
- Use all 2^n generalized AND gates with *n* inputs.
- Only one of them matches the input address.

Application (next lecture)
- Select a memory word for read/write.
- [Use address bits of instruction from IR.]

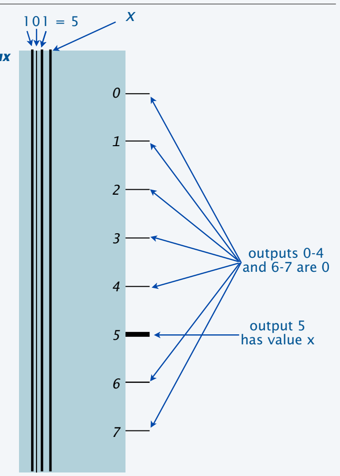
Example: 3-to-8 decoder



Another useful combinational circuit: demultiplexer (demux)

Demultiplexer
- *n* address inputs.
- 1 data input with value *x*.
- 2^n outputs.
- Addressed output has value *x*.
- All other outputs are 0.

Example: 3-to-8 demux



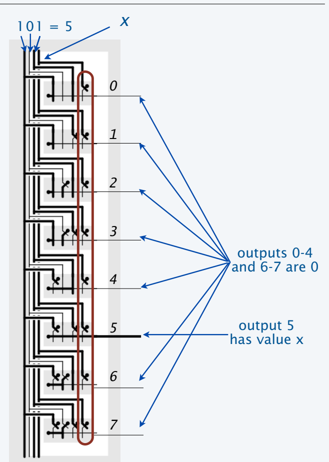
Another useful combinational circuit: demultiplexer (demux)

Demultiplexer
- *n* address inputs.
- 1 data input with value *x*.
- 2^n outputs.
- Addressed output has value *x*.
- All other outputs are 0.

Implementation
- Start with decoder.
- Add *AND x* to each gate.

Application (next lecture)
- Turn on control wires to implement instructions.
- [Use opcode bits of instruction in IR.]

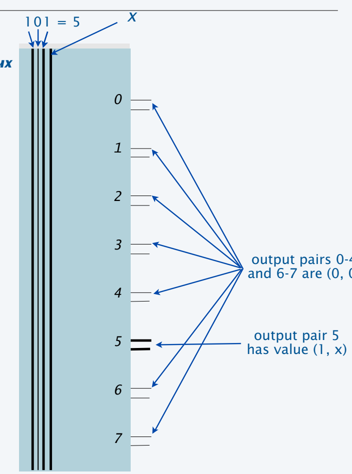
Example: 3-to-8 demux



Decoder/demux

Decoder/demux
- *n* address inputs.
- 1 data input with value *x*.
- 2^n output *pairs*.
- Addressed output *pair* has value (1, *x*).
- All other outputs are 0.

Example: 3-to-8 decoder/demux



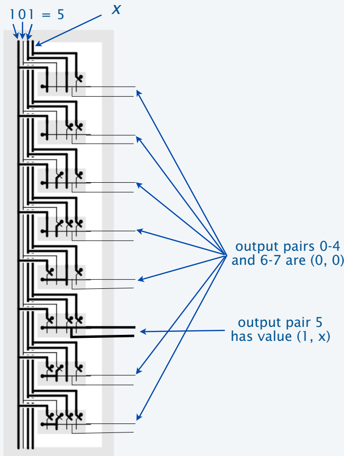
Decoder/demux

Decoder/demux
- n address inputs.
- 1 data input with value x.
- 2^n output *pairs*.
- Addressed output *pair* has value (1, x).
- All other outputs are 0.

Implementation
- Add decoder output to demux.

Application (next lecture)
- Access and control write of memory word
- [Use addr bits of instruction in IR.]

Example: 3-to-8 decoder/demux



Creating a digital circuit that computes a boolean function: majority

Use the truth table
- Identify rows where the function is 1.
- Use a generalized AND gate for each.
- OR the results together.

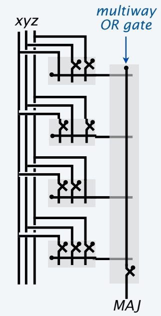
Example 1: Majority function

x	y	z	MAJ
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

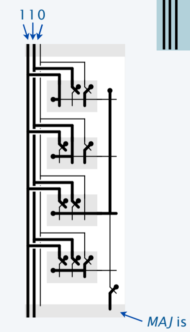
$MAJ = x'yz + xy'z + xyz' + xyz$

term *gate*

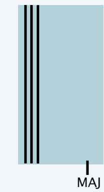
x'yz
xy'z
xyz'
xyz



majority circuit



example



Creating a digital circuit that computes a boolean function: odd parity

Use the truth table
- Identify rows where the function is 1.
- Use a generalized AND gate for each.
- OR the results together.

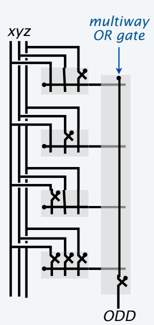
Example 2: Odd parity function

x	y	z	ODD
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

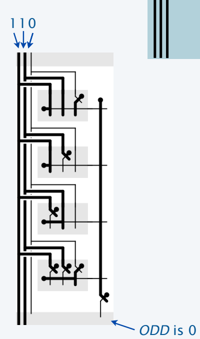
$ODD = x'y'z + x'yz' + xy'z' + xyz$

term *gate*

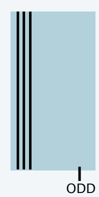
x'y'z
x'yz'
xy'z'
xyz



odd parity circuit



example



Combinational circuit design: Summary

Problem: Design a circuit that computes a given boolean function.

Ingredients
- OR gates.
- NOT gates.
- NOR gates.
- Wire.

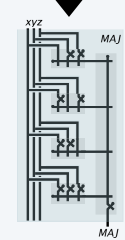
use to make generalized AND gates

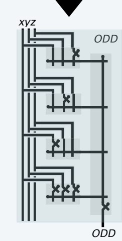
Method
- Step 1: Represent input and output with Boolean variables.
- Step 2: Construct truth table to define the function.
- Step 3: Identify rows where the function is 1.
- Step 4: Use a generalized AND for each and OR the results.

Bottom line (profound idea): Yields a circuit for ANY function.
Caveat: Circuit might be huge (stay tuned).

x	y	z	MAJ
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

x	y	z	ODD
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1





Pop quiz on combinational circuit design

Q. Design a circuit to implement XOR(x, y).

Pop quiz on combinational circuit design

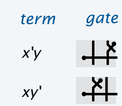
Q. Design a circuit to implement XOR(x, y).

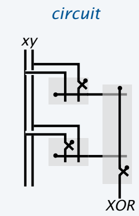
A. Use the truth table
- Identify rows where the function is 1.
- Use a generalized AND gate for each.
- OR the results together.

XOR function

x	*y*	*XOR*
0	0	0
0	1	1
1	0	1
1	1	0

XOR = x'y + xy'





Encapsulation

Encapsulation in hardware design mirrors familiar principles in software design
- Building a circuit from wires and switches is the *implementation*.
- Define a circuit by its inputs, controls, and outputs is the *API*.
- We control complexity by *encapsulating* circuits as we do with *ADTs*.

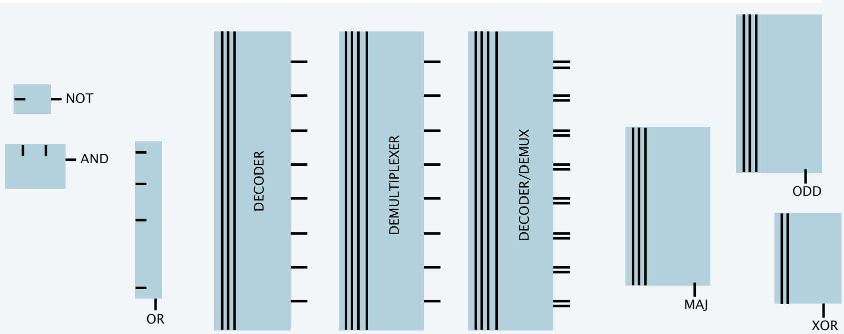


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Image sources

http://en.wikipedia.org/wiki/Claude_Shannon#/media/File:Claude_Elwood_Shannon_(1916-2001).jpg

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PART II: ALGORITHMS, MACHINES, and THEORY

19. Combinational Circuits

- Building blocks
- Boolean algebra
- Digital circuits
- **Adder circuit**
- Arithmetic/logic unit

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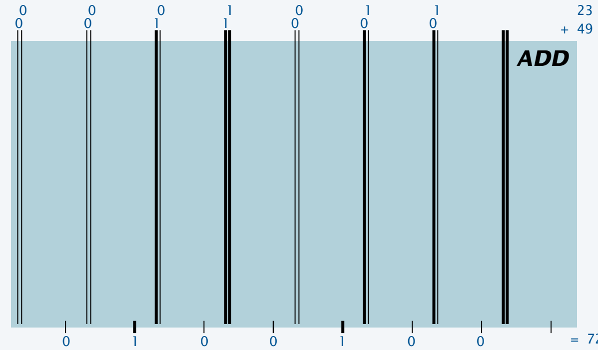
Let's make an adder circuit!

Adder
- Compute z = x + y for *n*-bit binary integers.
- 2*n* inputs.
- *n* outputs.
- Ignore overflow.

Example: 8-bit adder

carry out

0	0	1	1	0	1	1	1	0
	0	0	0	1	0	1	1	1
+	0	0	1	1	0	0	0	1
	0	1	0	0	1	0	0	0



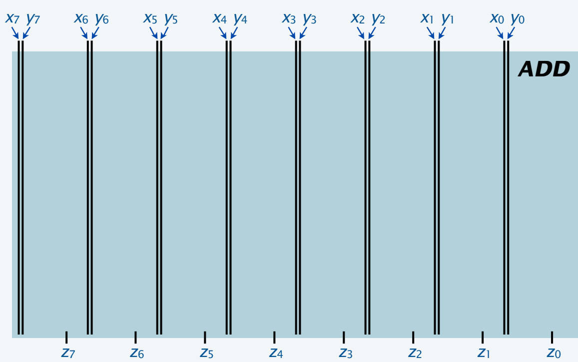
Let's make an adder circuit!

Adder
- Compute z = x + y for *n*-bit binary integers.
- 2*n* inputs.
- *n* outputs.
- Ignore overflow.

Example: 8-bit adder

carry out

c_8	c_7	c_6	c_5	c_4	c_3	c_2	c_1	0
	x_7	x_6	x_5	x_4	x_3	x_2	x_1	x_0
+	y_7	y_6	y_5	y_4	y_3	y_2	y_1	y_0
	z_7	z_6	z_5	z_4	z_3	z_2	z_1	z_0



Let's make an adder circuit!

Goal: $z = x + y$ for 8-bit integers.

Strawman solution: Build truth tables for each output bit.

c_8	c_7	c_6	c_5	c_4	c_3	c_2	c_1	0
	x_7	x_6	x_5	x_4	x_3	x_2	x_1	x_0
+	y_7	y_6	y_5	y_4	y_3	y_2	y_1	y_0
	z_7	z_6	z_5	z_4	z_3	z_2	z_1	z_0

8-bit adder truth table

x_7	x_6	x_5	x_4	x_3	x_2	x_1	x_0	y_7	y_6	y_5	y_4	y_3	y_2	y_1	y_0	c_4	z_7	z_6	z_5	z_4	z_3	z_2	z_1	z_0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0	0	1	1
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	0	1	1	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

$2^{16} = 65536$ rows!

Q. Not convinced this a bad idea?

A. 128-bit adder: 2^{256} rows >> # electrons in universe!

Let's make an adder circuit!

Goal: $z = x + y$ for 8-bit integers.

Do one bit at a time.

- Build truth table for carry bit.
- Build truth table for sum bit.

A surprise!

- Carry bit is MAJ.
- Sum bit is ODD.

c_8	c_7	c_6	c_5	c_4	c_3	c_2	c_1	0
	x_7	x_6	x_5	x_4	x_3	x_2	x_1	x_0
+	y_7	y_6	y_5	y_4	y_3	y_2	y_1	y_0
	z_7	z_6	z_5	z_4	z_3	z_2	z_1	z_0

carry bit

x_i	y_i	c_i	c_{i+1}	*MAJ*
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

sum bit

x_i	y_i	c_i	z_i	*ODD*
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	1	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

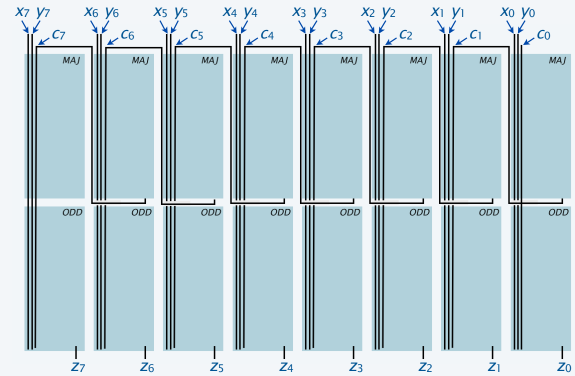
Let's make an adder circuit!

Goal: $z = x + y$ for 4-bit integers.

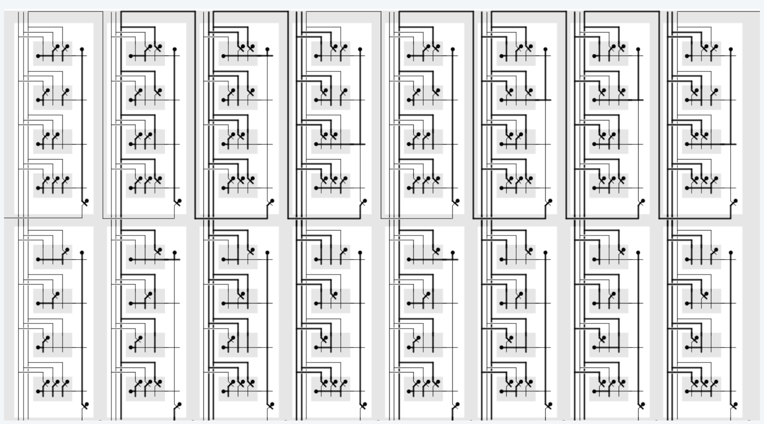
Do one bit at a time.

- Carry bit is MAJ.
- Sum bit is ODD.
- Chain 1-bit adders to "ripple" carries.

c_8	c_7	c_6	c_5	c_4	c_3	c_2	c_1	0
	x_7	x_6	x_5	x_4	x_3	x_2	x_1	x_0
+	y_7	y_6	y_5	y_4	y_3	y_2	y_1	y_0
	z_7	z_6	z_5	z_4	z_3	z_2	z_1	z_0



An 8-bit adder circuit



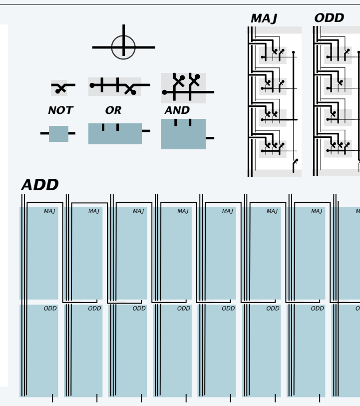
Layers of abstraction

Lessons for software design apply to hardware

- Interface describes behavior of circuit.
- Implementation gives details of how to build it.
- Exploit understanding of behavior at each level.

Layers of abstraction apply with a vengeance

- On/off.
- Controlled switch. [relay, pass transistor]
- Gates. [NOT, OR, AND]
- Boolean functions. [MAJ, ODD]
- Adder.
- Arithmetic/Logic unit (next).
- CPU (next lecture, stay tuned).



Vastly simplifies design of complex systems and enables use of new technology at any layer

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PART II: ALGORITHMS, MACHINES, and THEORY

19. Combinational Circuits

- Building blocks
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- **Arithmetic/logic unit**

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Next layer of abstraction: modules, busses, and control lines

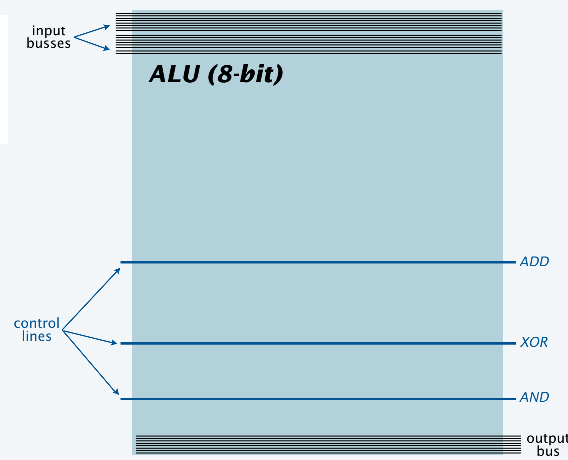
Basic design of our circuits
- Organized as *modules* (functional units of TOY: ALU, memory, register, PC, and IR).
- Connected by *busses* (groups of wires that propagate information between modules).
- Controlled by *control lines* (single wires that control circuit behavior).

Conventions
- Bus inputs are at the top, input connections are at the left.
- Bus outputs are at the bottom, output connections are at the right.
- Control lines are blue.

These conventions *make circuits easy to understand.*
(Like style conventions in coding.)

Arithmetic and logic unit (ALU) module

Ex. Three functions on 8-bit words
- Two input busses (arguments).
- One output bus (result).
- Three control lines.



Arithmetic and logic unit (ALU) module

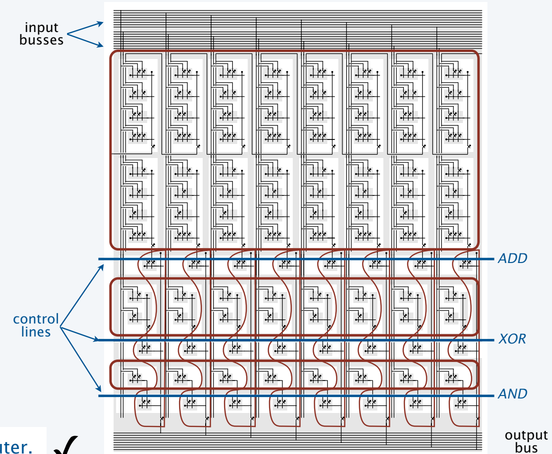
Ex. Three functions on 8-bit words
- Two input busses (arguments).
- One output bus (result).
- Three control lines.
- Left-right shifter circuits omitted (see book for details).

Implementation
- One circuit for each function.
- Compute all values in parallel.

Q. How do we select desired output?

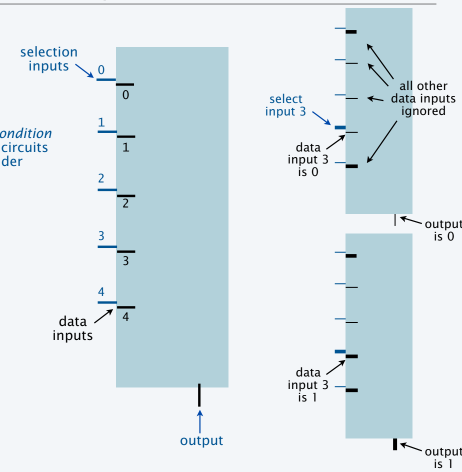
A. "One-hot muxes" (see next slide).

"Calculator" at the heart of your computer. ✓



A simple and useful combinational circuit: one-hot multiplexer

One-hot multiplexer
- *m* selection lines
- *m* data inputs
- 1 output.
- *At most one selection line is 1.* ← this is a *precondition* unlike other circuits we consider
- Output has value of selected input.



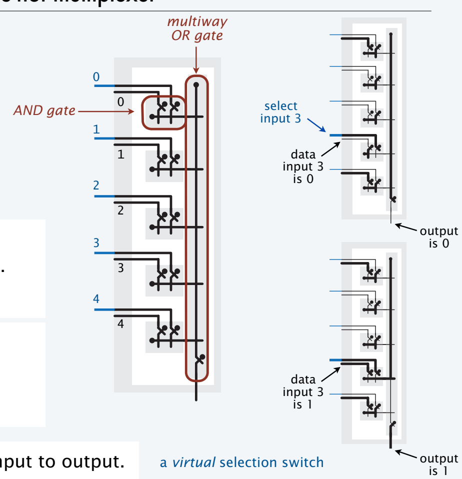
A simple and useful combinational circuit: one-hot multiplexer

One-hot multiplexer
- *m* selection lines
- *m* data inputs
- 1 output.
- *At most one selection line is 1.*
- Output has value of selected input.

Implementation
- AND corresponding selection and data inputs.
- OR all results (at most one is 1).

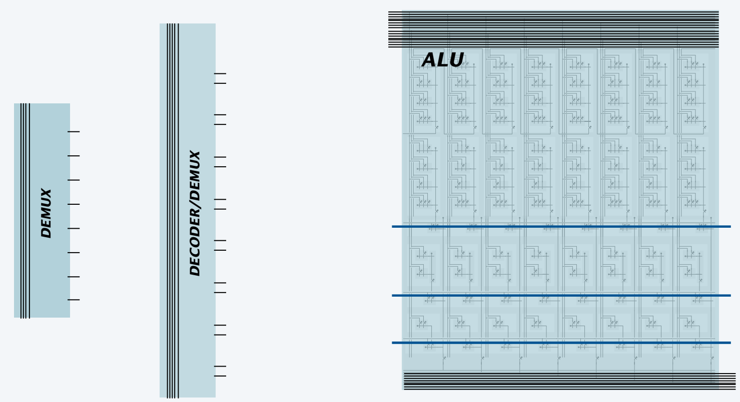
Applications
- Arithmetic-logic unit (previous slide).
- Main memory (next lecture).

Important to note. No direct connection from input to output.



a *virtual* selection switch

Summary: Useful combinational circuit modules



Next: Registers, memory, connections, and control.

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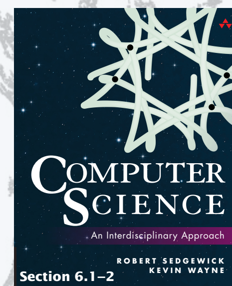

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19. Combinational Circuits