Introduction to Machine Learning - COS 324

Written Homework Assignment 4

Due date: The minute after 11:59pm of one week from announcement in class.

Electronic submissions only.

Note:

• Consulting with other students from this course is allowed. If you do so, clearly state whom you consulted with for each problem separately.
• Searching the internet or literature for solutions is prohibited.

I For this exercise we restrict ourselves to one dimensional functions, \( d = 1 \). Prove the equivalence of the two definitions of convexity shown in class. That is, we defined that \( f : K \mapsto \mathbb{R}^d \) is convex if and only if \( f((1-\alpha)x+\alpha y) \leq (1-\alpha)f(x)+\alpha f(y) \) for all \( x, y \in K \) and \( \alpha \in [0, 1] \). Show that \( f \) (assuming it is differentiable) is convex if and only if

\[
f(x) \geq f(y) + f'(y)(x - y)
\]

II Prove:

(a) The sum of convex functions is convex.
(b) Let \( f \) be \( \alpha_1 \)-strongly convex and \( g \) be \( \alpha_2 \)-strongly convex. Then \( f + g \) is \( (\alpha_1 + \alpha_2) \)-strongly convex.
(c) Let \( f \) be \( \beta_1 \)-smooth and \( g \) be \( \beta_2 \)-smooth. Then \( f + g \) is \( (\beta_1 + \beta_2) \)-smooth.

III BONUS Let \( f(x) : \mathbb{R}^d \mapsto \mathbb{R} \) be a convex differentiable function and \( \mathcal{K} \subseteq \mathbb{R}^d \) be a convex set. Prove that \( x^* \in \mathcal{K} \) is a minimizer of \( f \) over \( \mathcal{K} \) if and only if for any \( y \in K \) it holds that \( (y - x^*)^T \nabla f(x^*) \geq 0 \).