Introduction to Machine Learning - COS 324

Homework Assignment 1 Solutions

1. (Monty Hall problem.) You are given three curtains to choose from. Behind two of them sits goat while behind the third there is a state-of-the-art computer with a TPU. You do not know which one is which. You pick one curtain at random. Clearly, behind at least one of the two remaining curtains resides a goat. This curtain (or one of two) is revealed to you.

   (a) **Would you now change your choice in order to maximize your chance of getting a TPU? Prove your claim.**

   Choice: Yes, you should now change your choice.

   Proof:

   Let A be the event that you initially choose correctly, and A^C be the event that you initially choose incorrectly. Let W be the event that you get a TPU at the end, and W^C be the event that you do not.

   In your initial choice, since you pick one curtain at random and one of three choices is correct, $P(A) = \frac{1}{3}$ and $P(A^C) = \frac{2}{3}$.

   First we look at the probability that you win given that you do not switch doors. This is just the probability that you initially choose the correct door, which is $P(A) = \frac{1}{3}$.

   Next we look at the probability that you win given that you switch doors.

$$\begin{aligned}
P(W) &= P(W, A) + P(W, A^C) \\
&= P(W|A)P(A) + P(W|A^C)P(A^C) \\
&= (0)(\frac{1}{3}) + (1)(\frac{2}{3}) \\
&= \frac{2}{3}
\end{aligned}$$

   We have the first equality because the you either pick the correct door or you do not, so the probability of winning is the probability that you win and pick the correct door, or the probability that you win and do not pick the correct door. We have the second equality using conditional probability. We have the third equality because if you initially choose the correct door and switch then you have a probability 0 *of winning, and if you initially choose and incorrect door then the other incorrect door must be revealed, so (assuming that you choose to switch to the door that was not shown to have a goat) if you choose to switch then you have a probability* 1 *of winning.*

   Since the probability of winning is greater if you change your choice, you should change your choice.

2. Recall the weighted majority algorithm and its mistake bound.

   (a) Assume that you know the number of mistakes that the best expert is going to make. Call it $m^\star$. Describe changes to the WM algorithm such that the difference between the number of mistakes it makes, denoted m, and twice that of the best expert in indsight, is bounded by

$$m - 2m^\star \le 4\sqrt{m^\star \log n} + 4\log n \quad .$$

Proof. We know from lecture slides that number of mistakes made by the WM algorithm satisfies the following (for $0 \le \eta \le 1/2$)

$$m - 2m^* \le 2\eta m^* + \frac{2\log n}{\eta}$$

We wish to get smallest upper bound. The upper bound is minimized at $\eta = \sqrt{\frac{\log n}{m^*}}$ (can be verified by setting derivative to 0 and checking that second derivative is positive). We need to deal with 2 cases

i. $m^* > 4\log n$: We can pick $\eta = \sqrt{\frac{\log n}{m^*}} < \frac{1}{2}$. The upper bound in this case is

$$2\sqrt{\frac{\log n}{m^*}}m^* + 2\log n\sqrt{\frac{m^*}{\log n}} = 4\sqrt{m^*\log n} < 4\sqrt{m^*\log n} + 4\log n$$

ii. $m^* \le 4\log n$: We can't pick the same value of η as before as the upper bound is only valid for $\eta \le 1/2$. So we pick $\eta = 1/2$. The upper bound in this case is

$$m^* + 4\log n = \sqrt{m^*}\sqrt{m^*} + 4\log n \le 2\sqrt{m^*\log n} + 4\log n < 4\sqrt{m^*\log n} + 4\log n$$