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3.2 Binary Search Trees

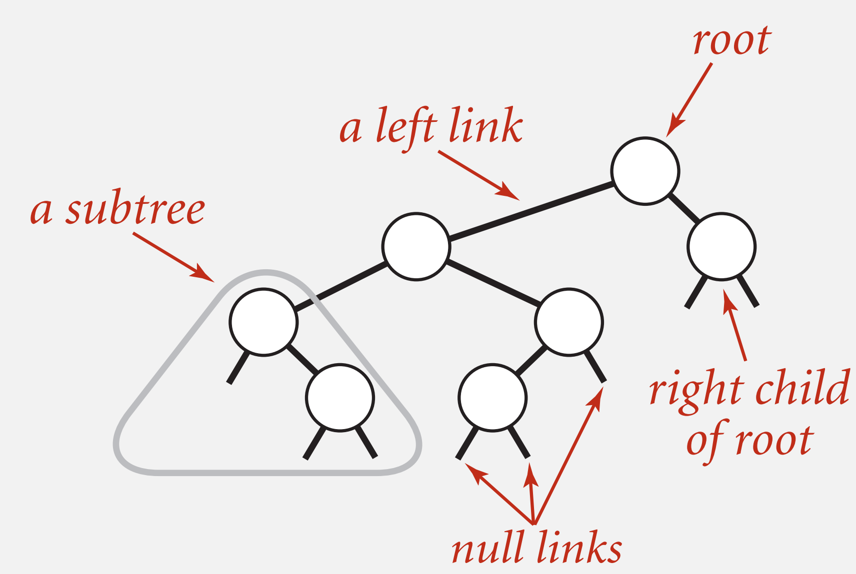
- *BSTs*
- *ordered operations*
- *iteration*
- *deletion*

Binary search trees

Definition. A BST is a binary tree in symmetric order.

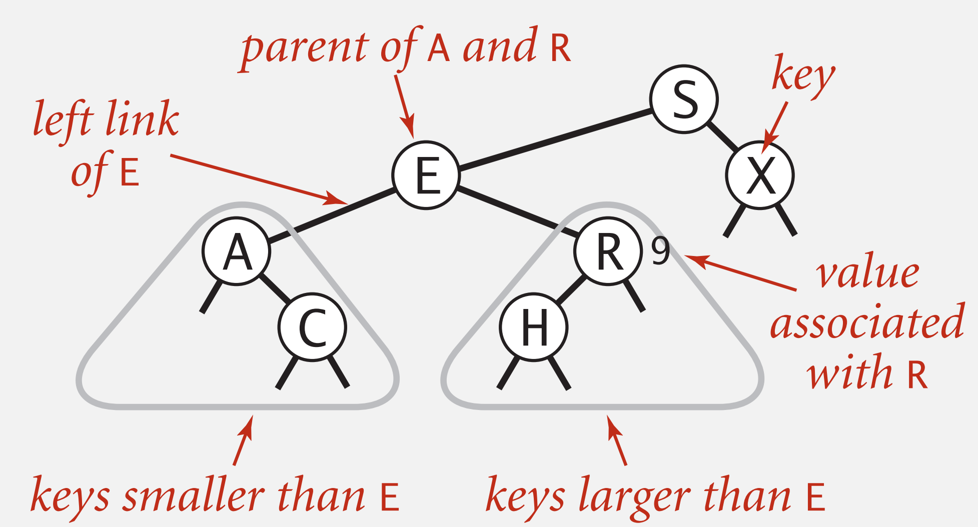
A binary tree is either:

- Empty.
- Two disjoint binary trees (left and right).



Symmetric order. Each node has a key, and every node's key is:

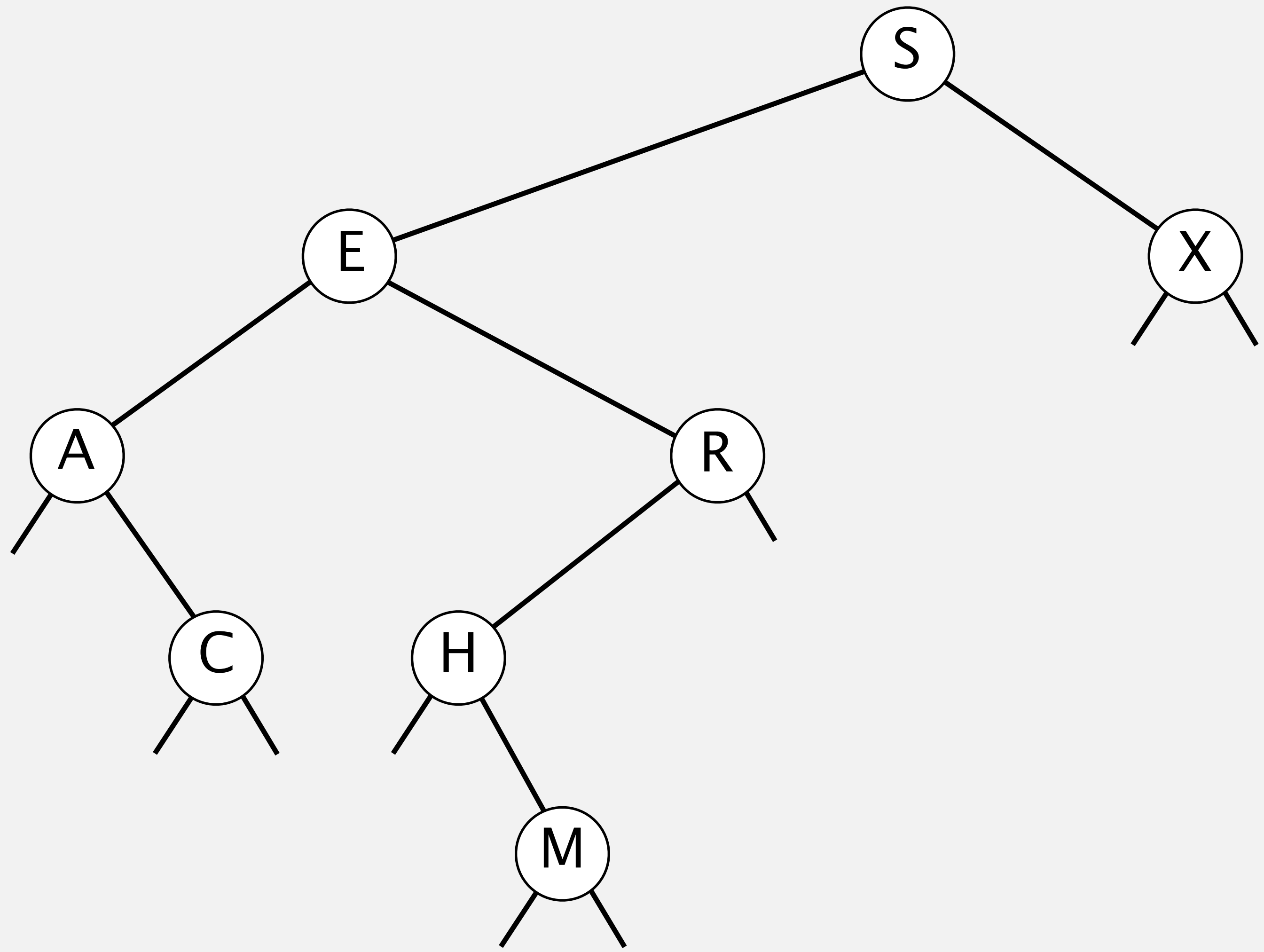
- Larger than all keys in its left subtree.
- Smaller than all keys in its right subtree.



Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

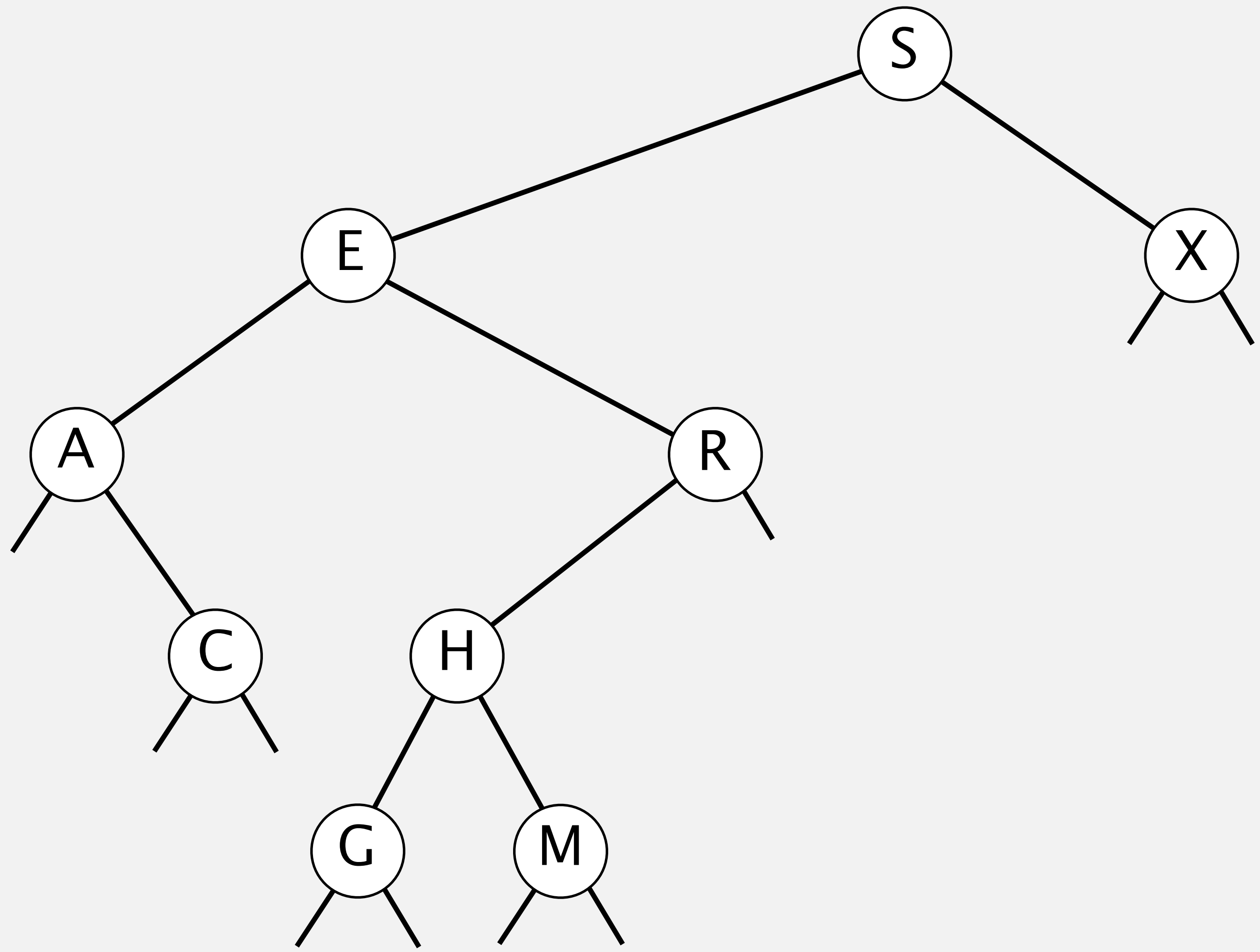
successful search for H



Binary search tree demo

Insert. If less, go left; if greater, go right; if `null`, insert.

insert G



BST representation in Java

Java definition. A BST is a reference to a root `Node`.

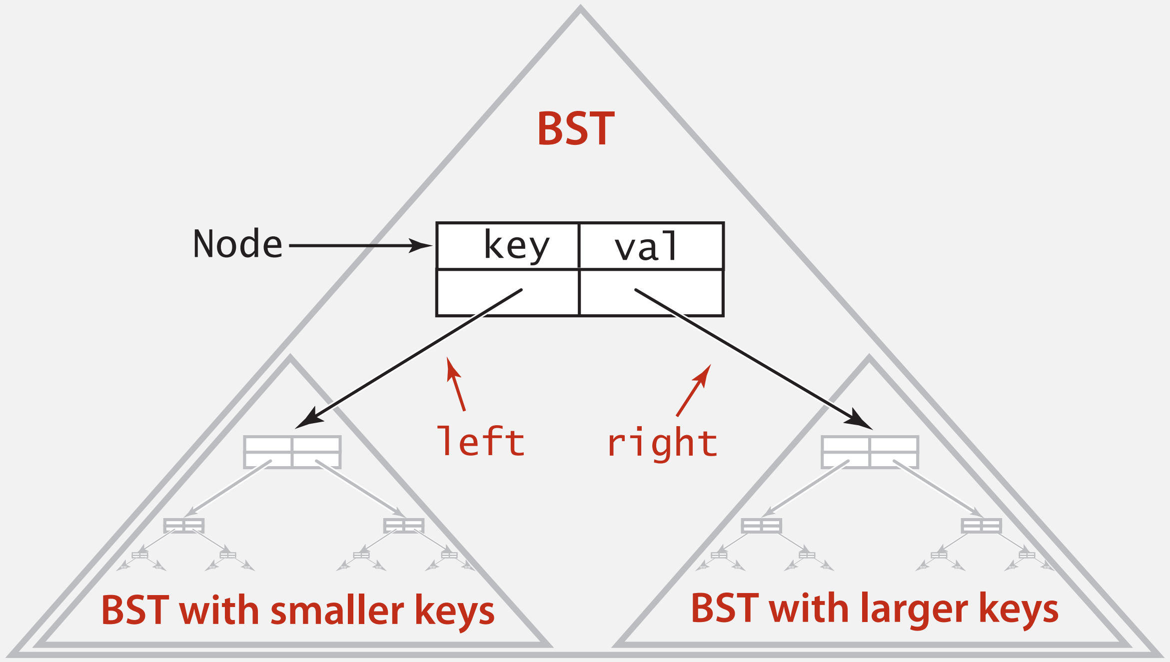
A `Node` is composed of four fields:

- A `Key` and a `Value`.
- A reference to the left and right subtree.

smaller keys larger keys

```
private class Node
{
   private Key key;
   private Value val;
   private Node left, right;

   public Node(Key key, Value val)
   {
      this.key = key;
      this.val = val;
   }
}
```

Key and Value are generic types; Key is Comparable



Binary search tree

BST implementation (skeleton)

```
public class BST<Key extends Comparable<Key>, Value>
{
   private Node root;                    root of BST

   private class Node
   { /* see previous slide */  }

   public void put(Key key, Value val)
   { /* see next slide */  }

   public Value get(Key key)
   { /* see next slide */  }

   public Iterable<Key> iterator()
   { /* see slides in next section */  }

   public void delete(Key key)
   { /* see textbook */  }

}
```

BST search: Java implementation

Get. Return value corresponding to given key, or `null` if no such key.

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```

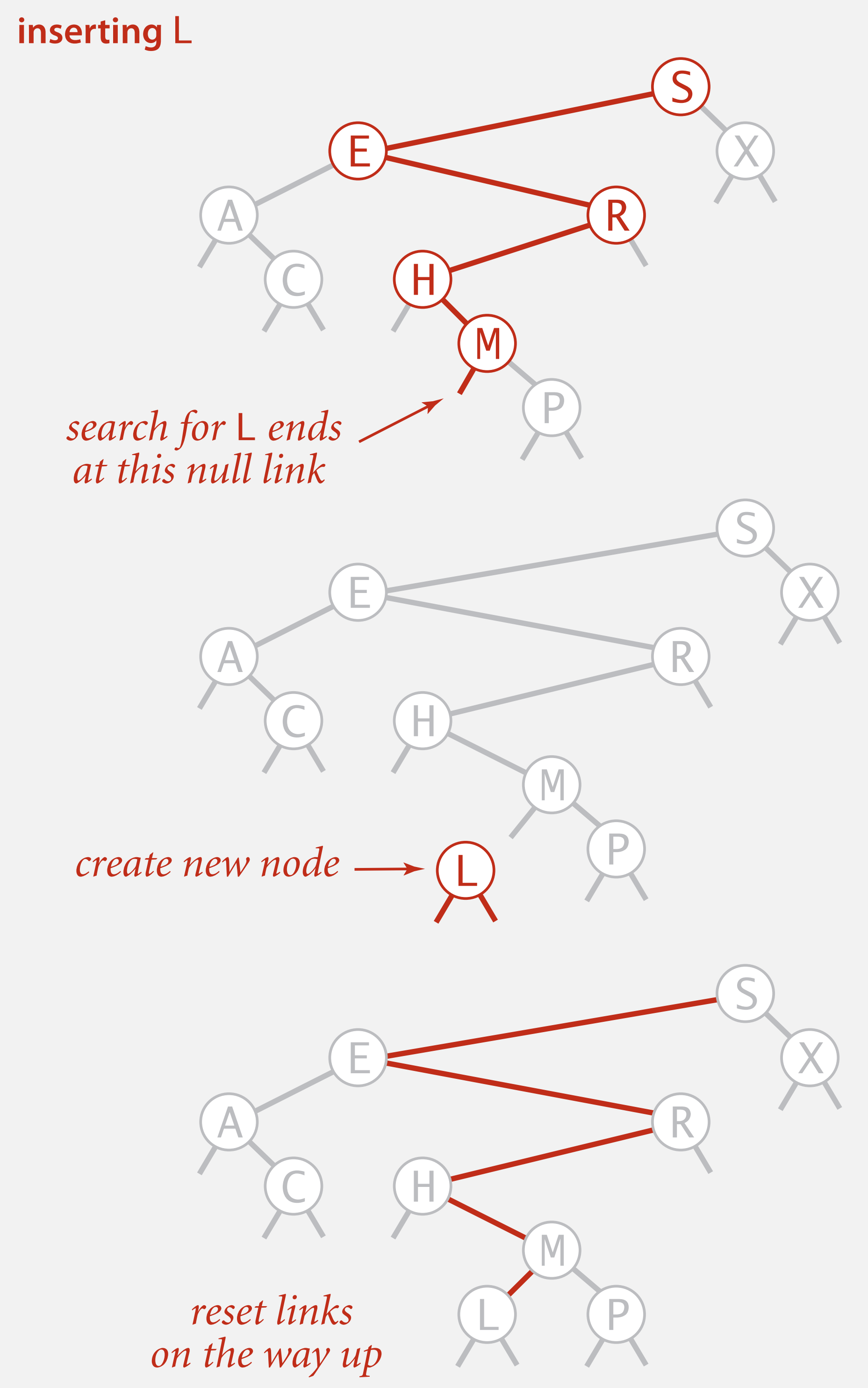
Cost. Number of compares = 1 + depth of node.

BST insert

Put. Associate value with key.

Search for key, then two cases:

- Key in tree $\Rightarrow$ reset value.
- Key not in tree $\Rightarrow$ add new node.



Insertion into a BST

BST insert: Java implementation

Put. Associate value with key.

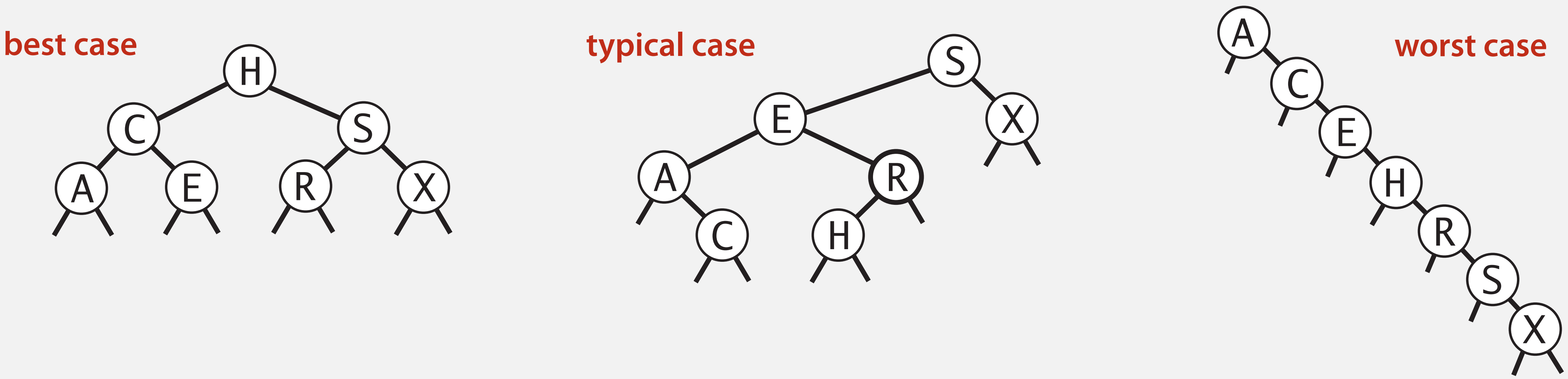
```
public void put(Key key, Value val)
{  root = put(root, key, val);  }

private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) x.left  = put(x.left,  key, val);
   else if (cmp  > 0) x.right = put(x.right, key, val);
   else if (cmp == 0) x.val   = val;
   return x;
}
```

⚠ **Warning: concise but tricky code; read carefully!**

Cost. Number of compares = 1 + depth of node.

Tree shape

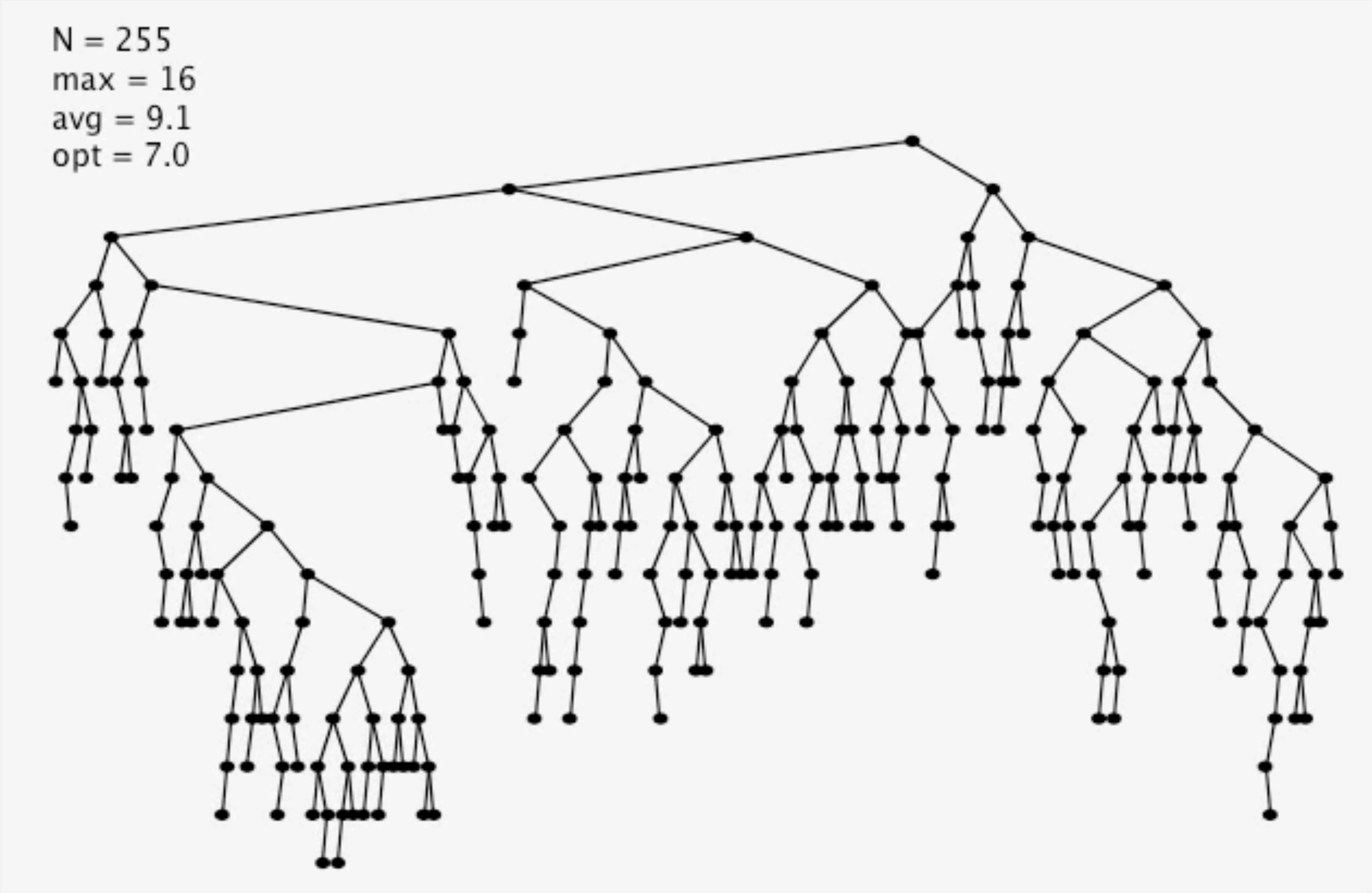
- Many BSTs correspond to same set of keys.
- Number of compares for search/insert = 1 + depth of node.



Bottom line. Tree shape depends on order of insertion.

BST insertion: random order visualization

Ex. Insert keys in random order.



Binary search trees: quiz 1

What is the expected number of compares to sort n distinct keys using the following sorting algorithm?

1. **Shuffle** the keys.
2. **Insert** the keys into a BST, one at a time.
3. Do an **inorder traversal** of the BST.

A. $\sim n \lg n$

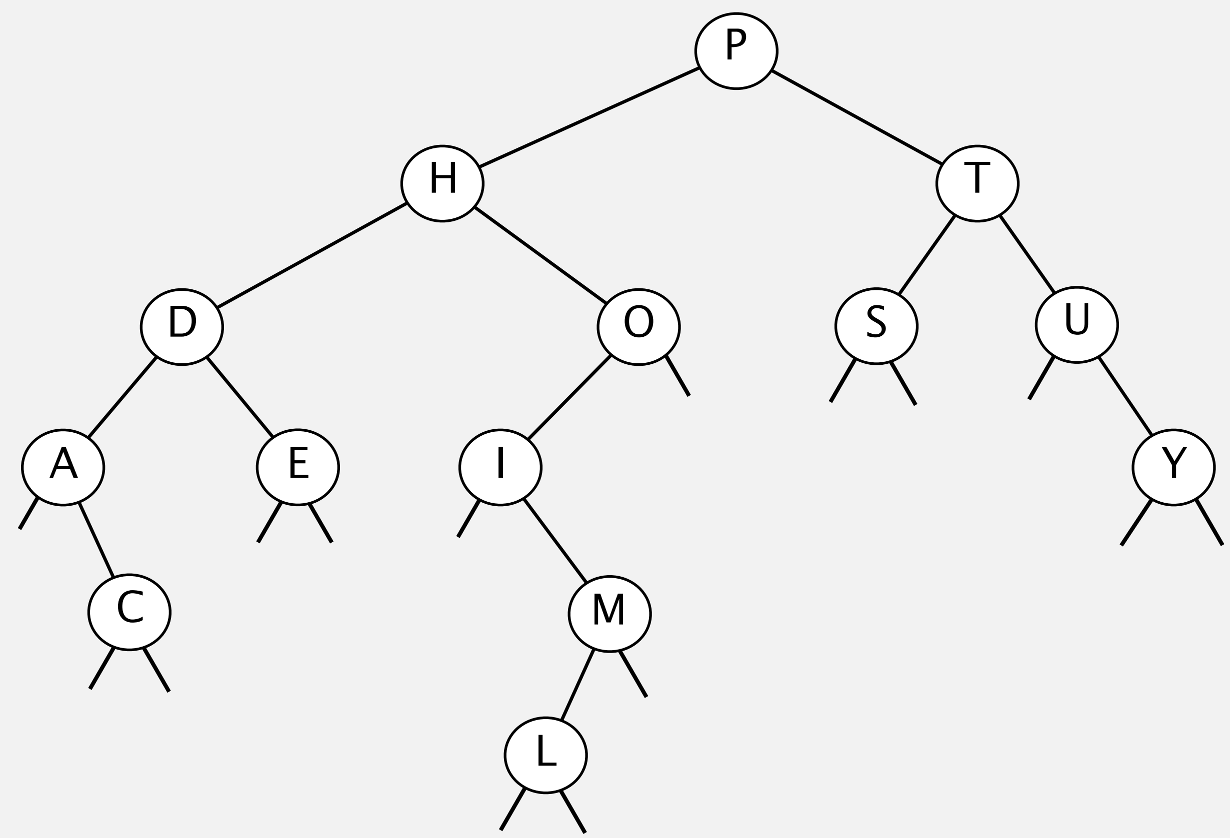
B. $\sim n \ln n$

C. $\sim 2\, n \lg n$

D. $\sim 2\, n \ln n$

Correspondence between BSTs and quicksort partitioning

0	1	2	3	4	5	6	7	8	9	10	11	12	13
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
H	L	E	A	D	O	M	C	I	P	T	Y	U	S
D	C	E	A	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y



Remark. Correspondence is 1–1 if array has no duplicate keys.

BSTs: mathematical analysis

Proposition. If n distinct keys are inserted into a BST in random order, the expected number of compares for a search/insert is $\sim 2 \ln n$.

Pf. 1–1 correspondence with quicksort partitioning.

Proposition. [Reed, 2003] If n distinct keys are inserted into a BST in random order, the expected height is $\sim 4.311 \ln n$.

expected depth of
function-call stack in quicksort

How Tall is a Tree?

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ABSTRACT

Let H_n be the height of a random binary search tree on n nodes. We show that there exists constants $\alpha = 4.31107\ldots$ and $\beta = 1.95\ldots$ such that $\mathrm{E}(H_n) = \alpha \log n - \beta \log\log n + O(1)$, We also show that $\mathrm{Var}(H_n) = O(1)$.

But... Worst-case height is $n - 1$.

[exponentially small chance when keys are inserted in random order]

ST implementations: summary

implementation	guarantee		average case		operations on keys
	search	insert	search hit	insert	
sequential search (unordered list)	n	n	n	n	`equals()`
binary search (ordered array)	$\log n$	n	$\log n$	n	`compareTo()`
BST	n	n	$\log n$	$\log n$	`compareTo()`

Why not shuffle to ensure a (probabilistic) guarantee of log n?

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3.2 Binary Search Trees

- BSTs
- *iteration*
- ordered operations
- deletion

Binary search trees: quiz 2

In which order does `traverse(root)` print the keys in the BST?

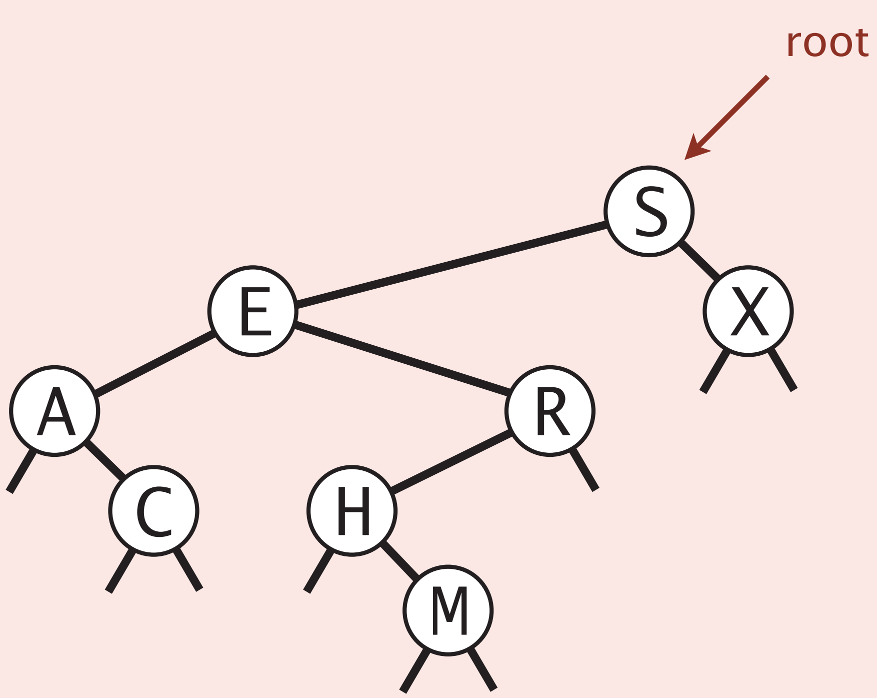
```
private void traverse(Node x)
{
   if (x == null) return;
   traverse(x.left);
   StdOut.println(x.key);
   traverse(x.right);
}
```

A. A C E H M R S X

B. S E A C R H M X

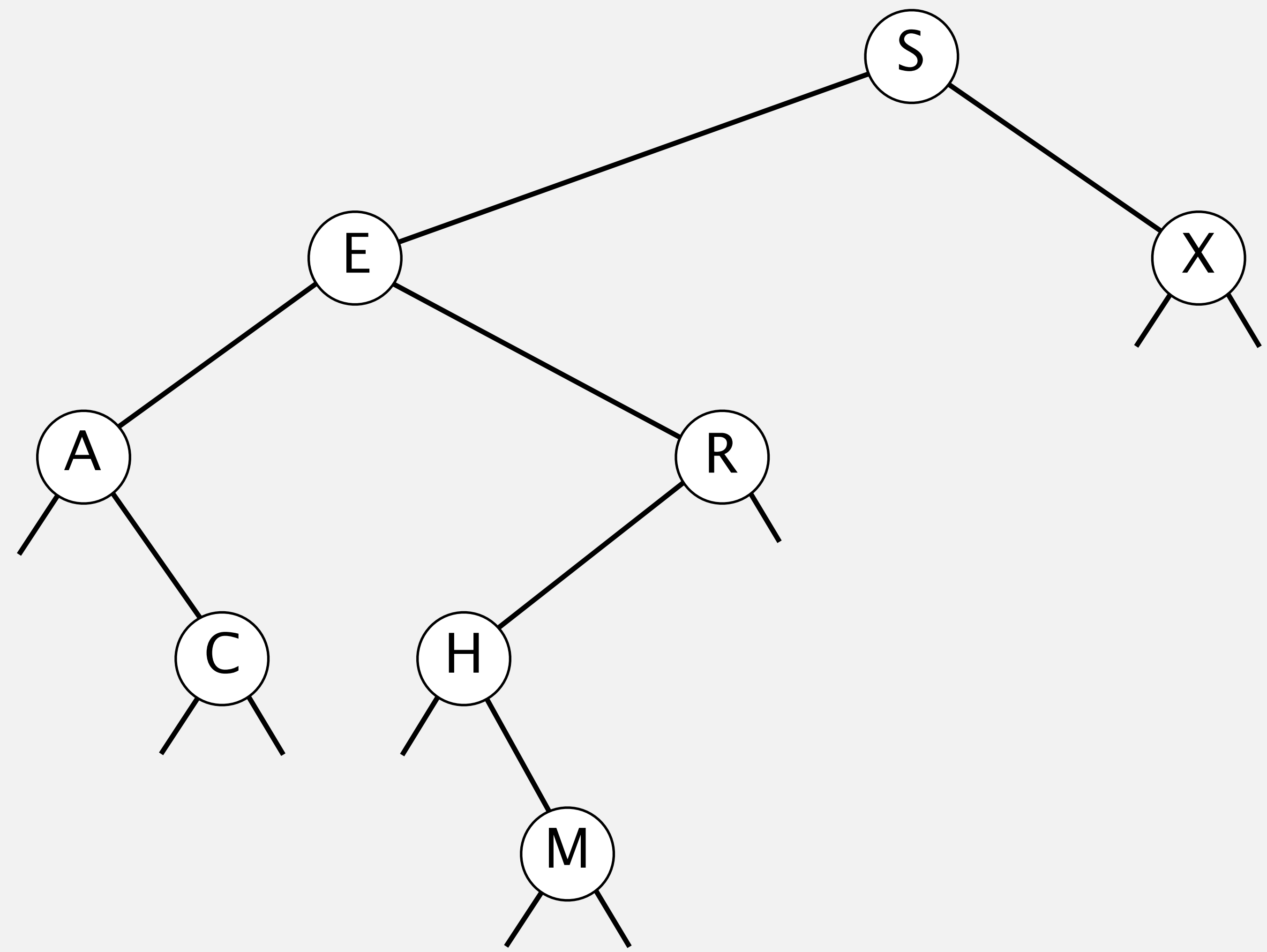
C. C A M H R E X S

D. S E X A R C H M



Inorder traversal

```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
               done M
            done H
         print R
         done R
      done E
   print S
   inorder(X)
      print X
      done X
   done S
```



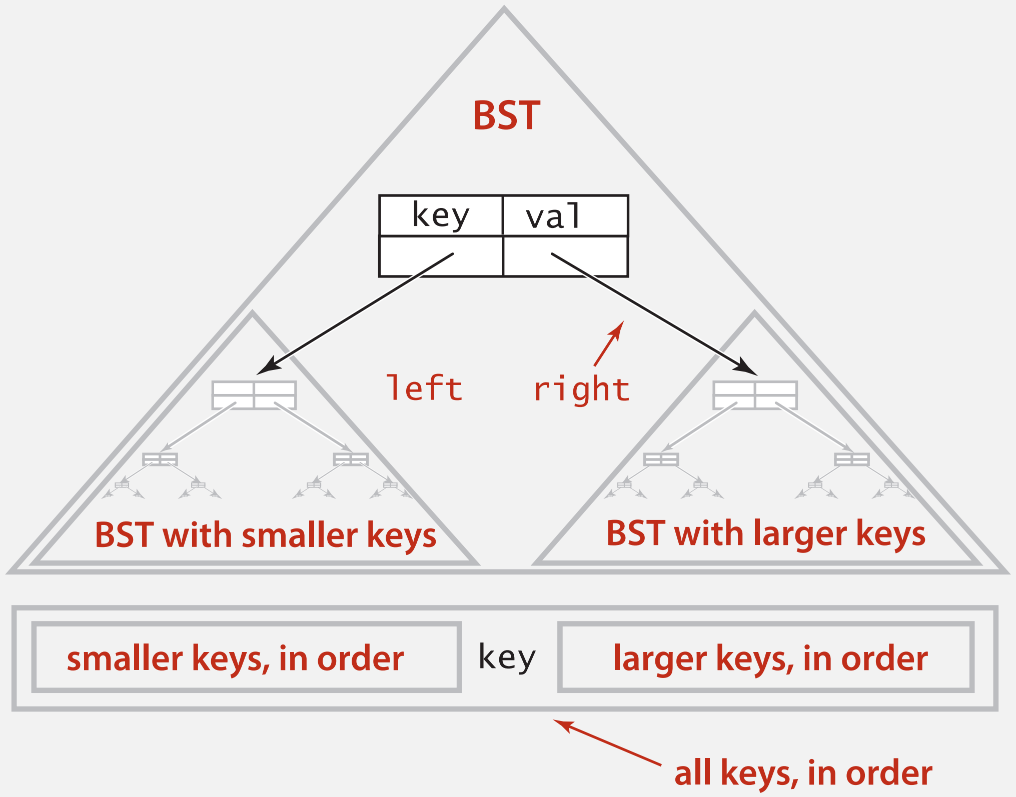
output: A C E H M R S X

Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```
public Iterable<Key> keys()
{
   Queue<Key> q = new Queue<Key>();
   inorder(root, q);
   return q;
}
```

```
private void inorder(Node x, Queue<Key> q)
{
   if (x == null) return;
   inorder(x.left, q);
   q.enqueue(x.key);
   inorder(x.right, q);
}
```



Property. Inorder traversal of a BST yields keys in ascending order.

Running time

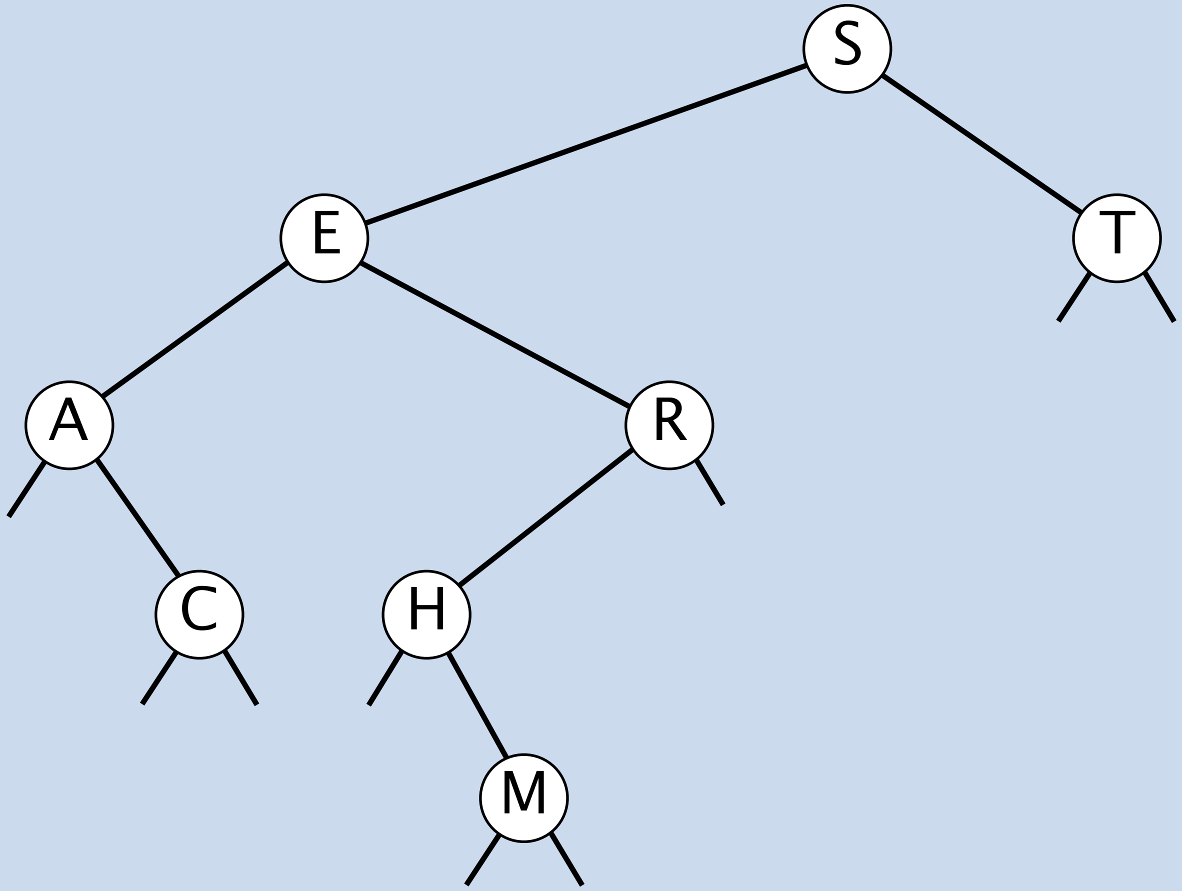
Property. Inorder traversal of a BST takes linear time.

Silicon Valley, Season 4, Episode 5

Level-Order Traversal

Level-order traversal of a binary tree.

- Process root.
- Process children of root, from left to right.
- Process grandchildren of root, from left to right.
- ...

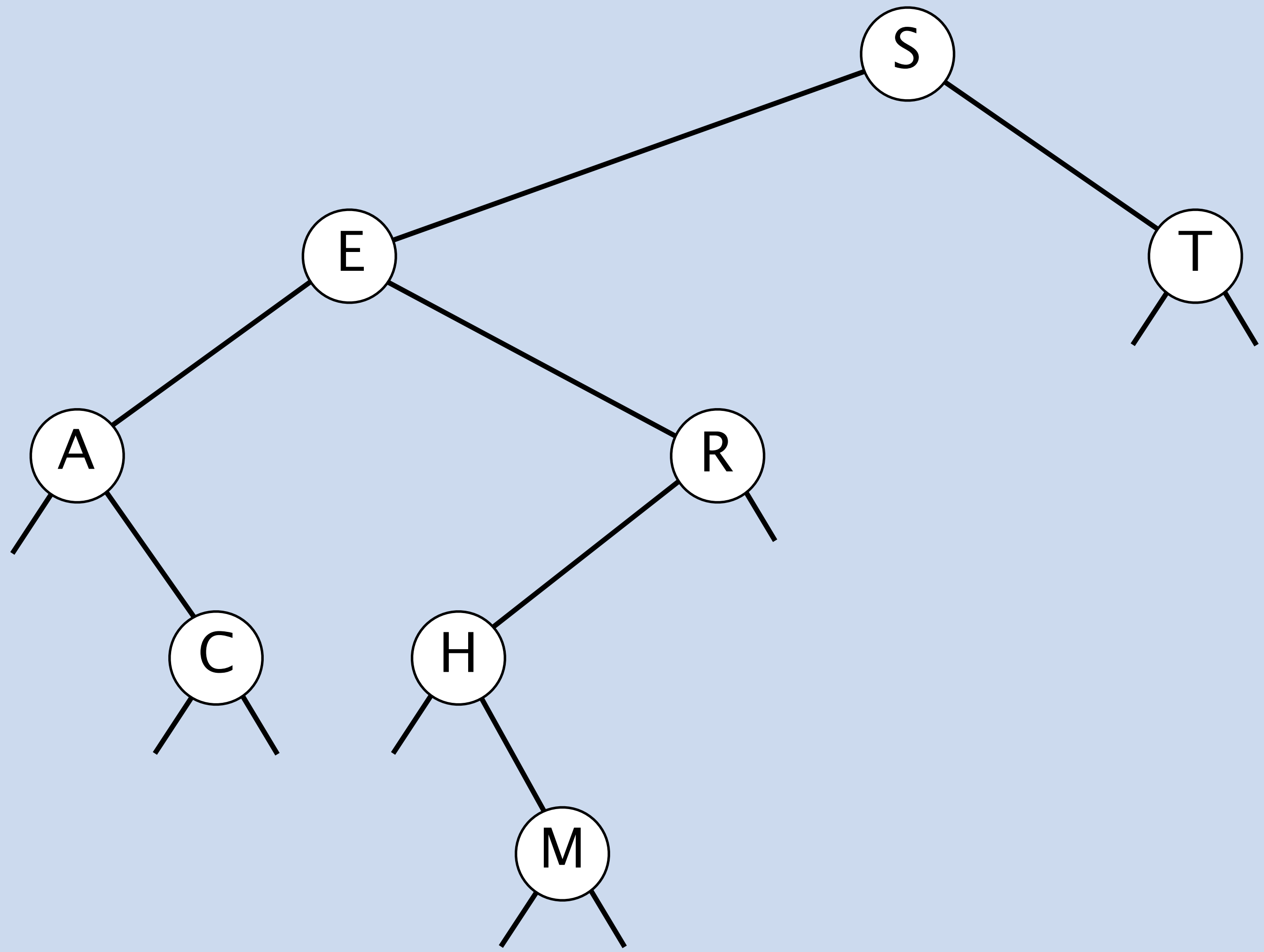


level-order traversal: S E T A R C H M

LEVEL-ORDER TRAVERSAL

Q2. Given the level-order traversal of a BST, how to (uniquely) reconstruct?

Ex. ~~S~~ ~~E~~ ~~T~~ ~~A~~ ~~R~~ ~~C~~ ~~H~~ ~~M~~



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3.2 Binary Search Trees

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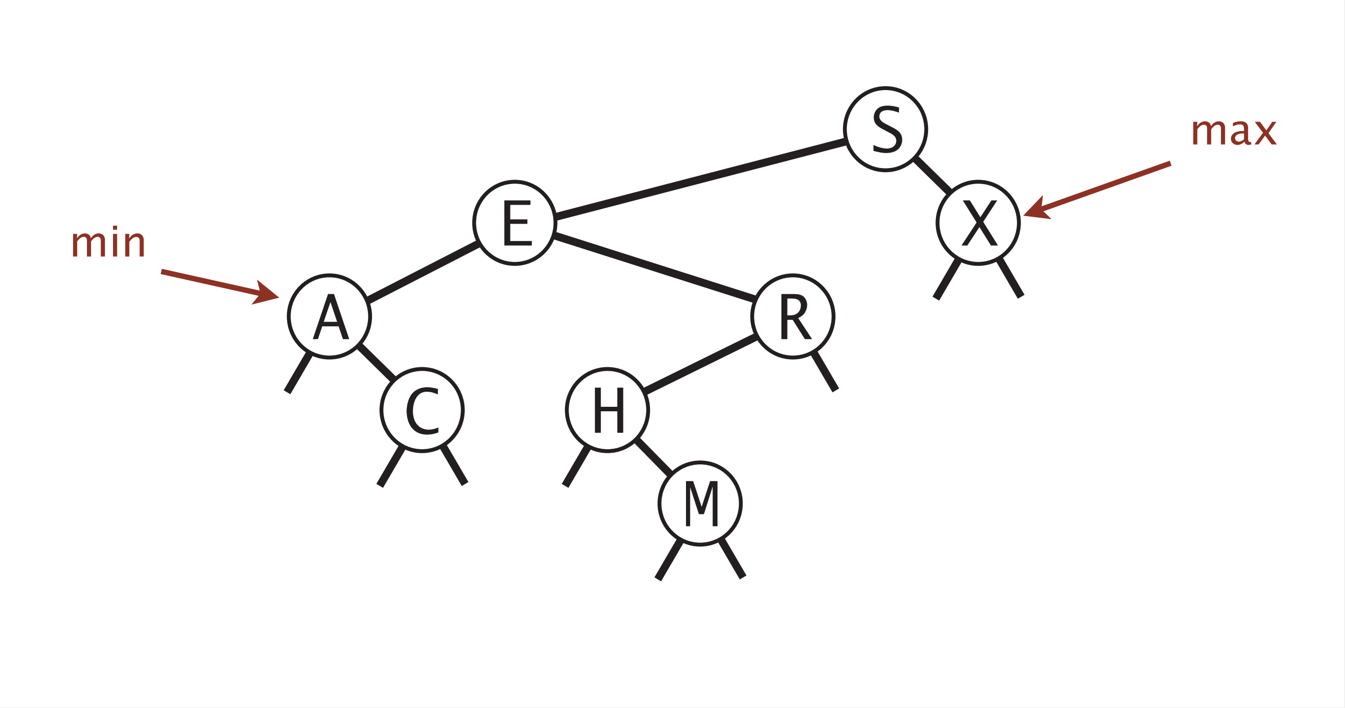
Minimum and maximum

Minimum. Smallest key in BST.

Maximum. Largest key in BST.

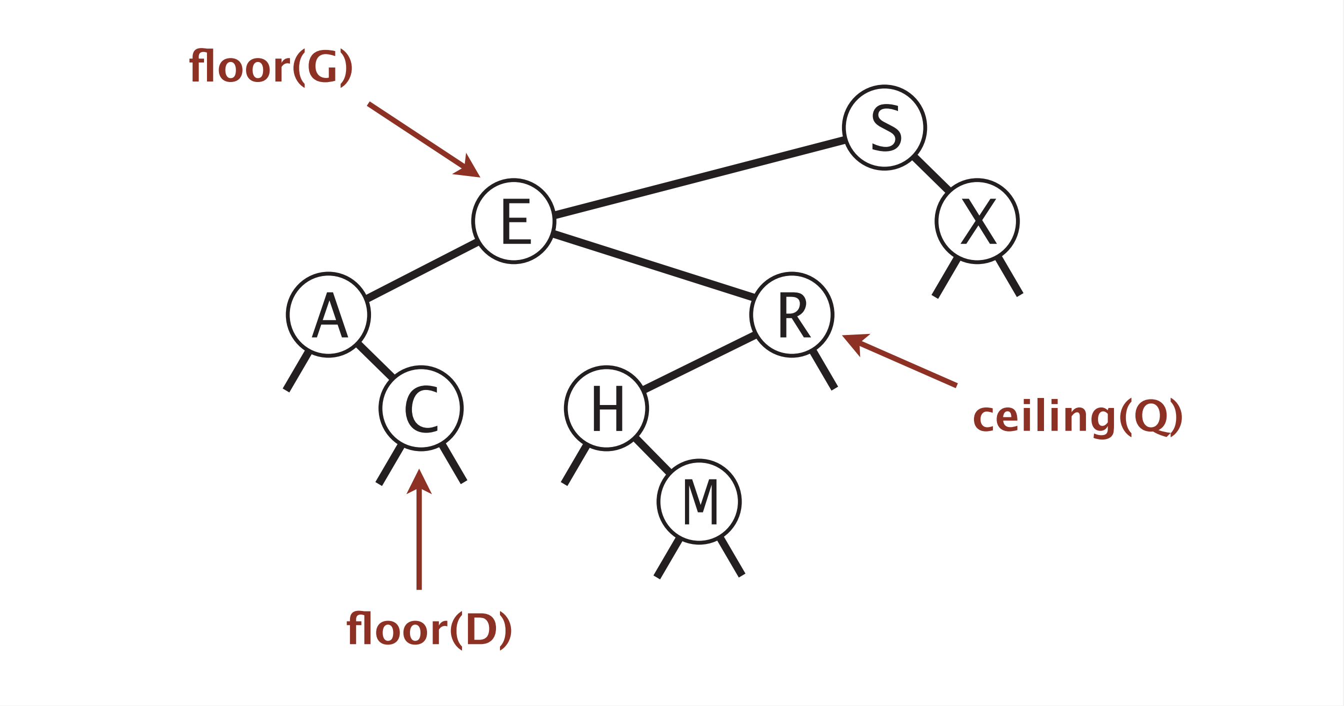


Q. How to find the min / max?

Floor and ceiling

Floor. Largest key in BST $\leq$ query key.

Ceiling. Smallest key in BST $\geq$ query key.



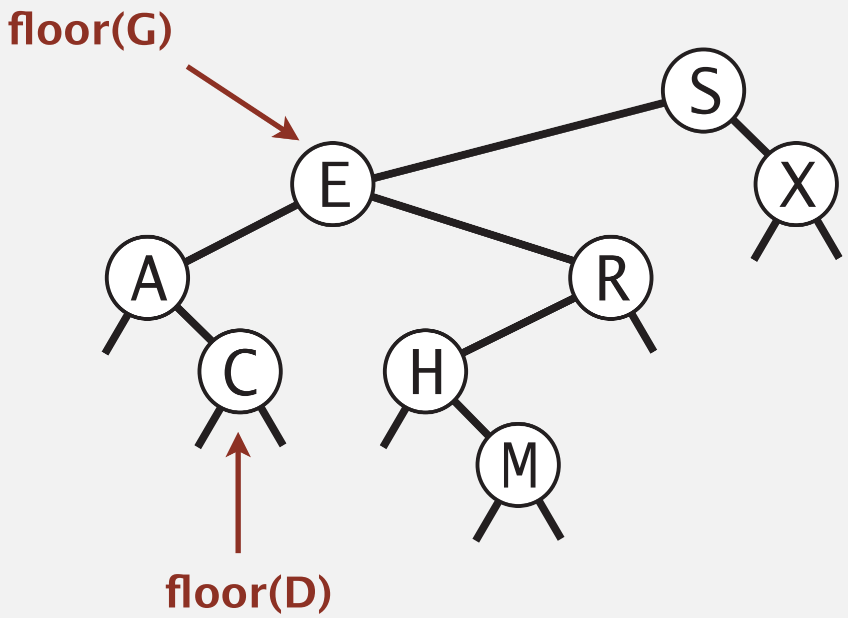
Q. How to find the floor / ceiling?

Computing the floor

Floor. Largest key in BST $\leq k$?

Key idea.

- To compute `floor(key)`, search for key.
- On search path, must encounter `floor(key)` and `ceiling(key)`. Why?



Computing the floor

```
public Key floor(Key key)
{  return floor(root, key, null);  }

private Key floor(Node x, Key key, Key best)
{
   if (x == null) return best;
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) return floor(x.left, key, best);
   else if (cmp  > 0) return floor(x.right, key, x.key);
   else if (cmp == 0) return x.key;
}
```

key in node is too big
(so look in left subtree)

key in node is best candidate for floor
(but maybe better one in right subtree)

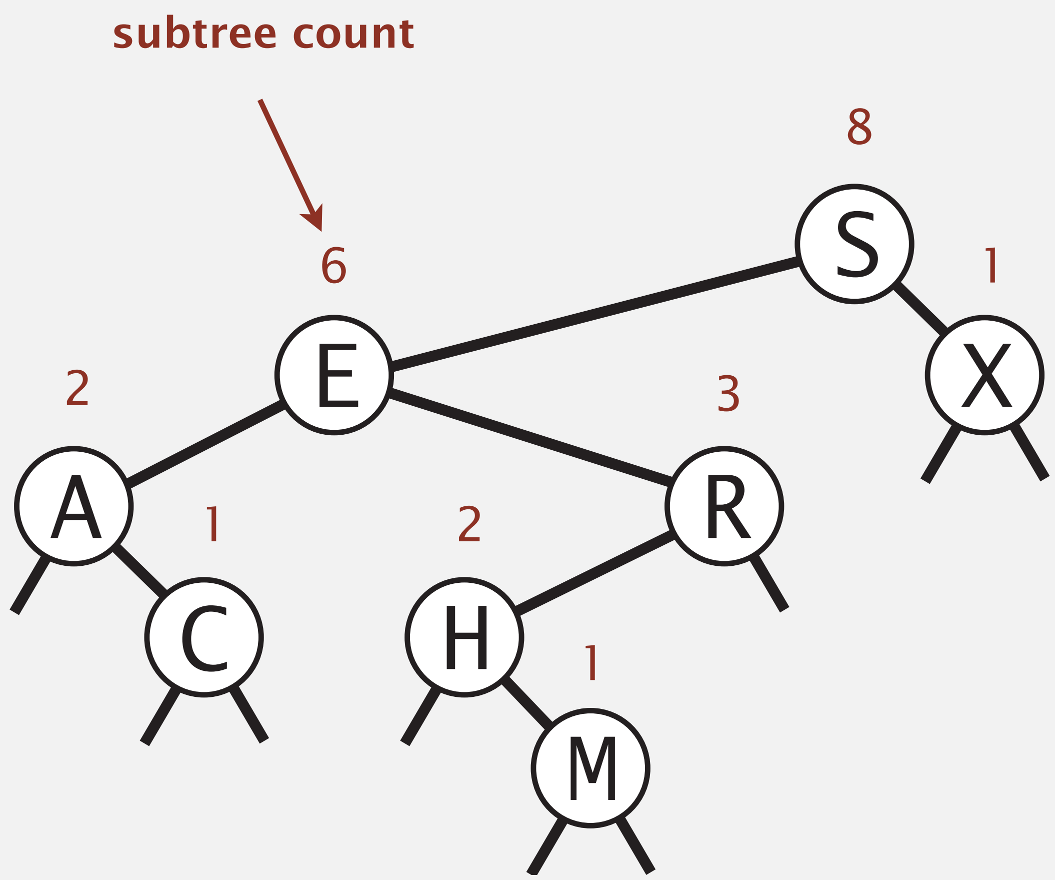
Rank and select

Rank. How many keys < *key* ?

Select. Key of rank *k*.

Q. How to implement `rank()` and `select()` efficiently for BSTs?

A. In each node, store the number of nodes in its subtree.



BST implementation: subtree counts

```
private class Node
{
   private Key key;
   private Value val;
   private Node left;
   private Node right;
   private int count;
}
```

number of nodes in subtree

```
public int size()
{  return size(root);  }

private int size(Node x)
{
   if (x == null) return 0;
   return x.count;
}
```

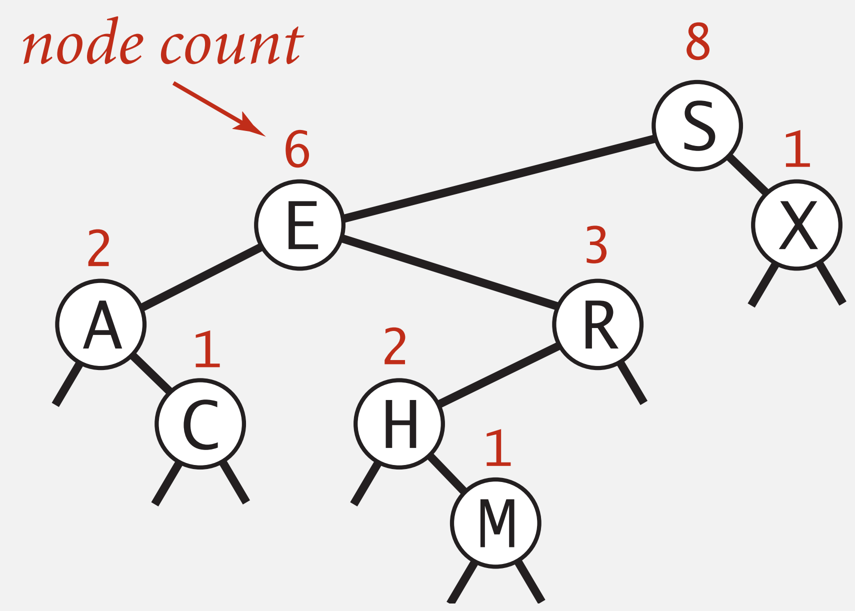
ok to call when x is null

```
private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val, 1);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) x.left  = put(x.left,  key, val);
   else if (cmp  > 0) x.right = put(x.right, key, val);
   else if (cmp == 0) x.val = val;

   x.count = 1 + size(x.left) + size(x.right);

   return x;
}
```

initialize subtree count to 1

Computing the rank

Rank. How many keys < *key* ?



Case 1. [*key* < key in node]

- Keys in left subtree? *count*
- Key in node? 0
- Keys in right subtree? 0

Case 2. [*key* > key in node]

- Keys in left subtree? *all*
- Key in node. 1
- Keys in right subtree? *count*

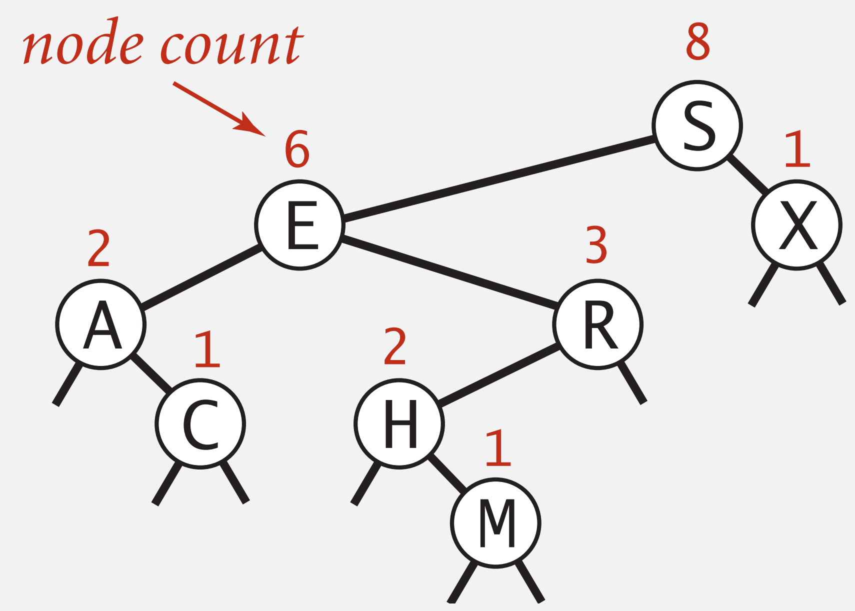
Case 3. [*key* = key in node]

- Keys in left subtree? *count*
- Key in node. 0
- Keys in right subtree? 0

Rank

Rank. How many keys < *key* ?

Easy recursive algorithm (3 cases!)




```
public int rank(Key key)
{  return rank(key, root);  }

private int rank(Key key, Node x)
{
   if (x == null) return 0;
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) return rank(key, x.left);
   else if (cmp  > 0) return 1 + size(x.left) + rank(key, x.right);
   else if (cmp == 0) return size(x.left);
}
```

BST: ordered symbol table operations summary

	sequential search	binary search	BST
search	n	$\log n$	h
insert	n	n	h
min / max	n	1	h
floor / ceiling	n	$\log n$	h
rank	n	$\log n$	h
select	n	1	h
ordered iteration	$n \log n$	n	n

h = height of BST

order of growth of running time of ordered symbol table operations

ST implementations: summary

implementation	guarantee		average case		ordered ops?	key interface
	search	insert	search hit	insert		
sequential search (unordered list)	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	$\log n$	n	✔	`compareTo()`
BST	n	n	$\log n$	$\log n$	✔	`compareTo()`
red-black BST	$\boxed{\log n}$	$\boxed{\log n}$	$\log n$	$\log n$	✔	`compareTo()`

Next week. Guarantee logarithmic performance for all operations.