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2.2 MERGESORT

- *mergesort*
- *bottom-up mergesort*
- *sorting complexity*
- *divide-and-conquer*

Two classic sorting algorithms: mergesort and quicksort

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20^{th} century in science and engineering.

Mergesort. [this lecture]

OCaml

...

Quicksort. [next lecture]

OCaml

...

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2.2 Mergesort

- mergesort
- bottom-up mergesort
- sorting complexity
- divide-and-conquer

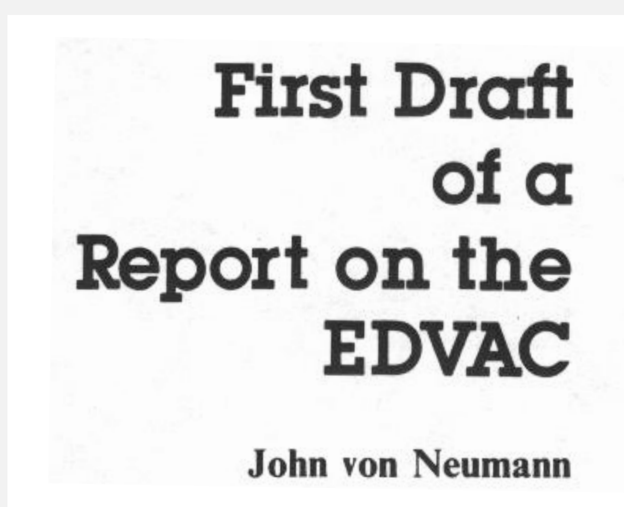
Mergesort

Basic plan.

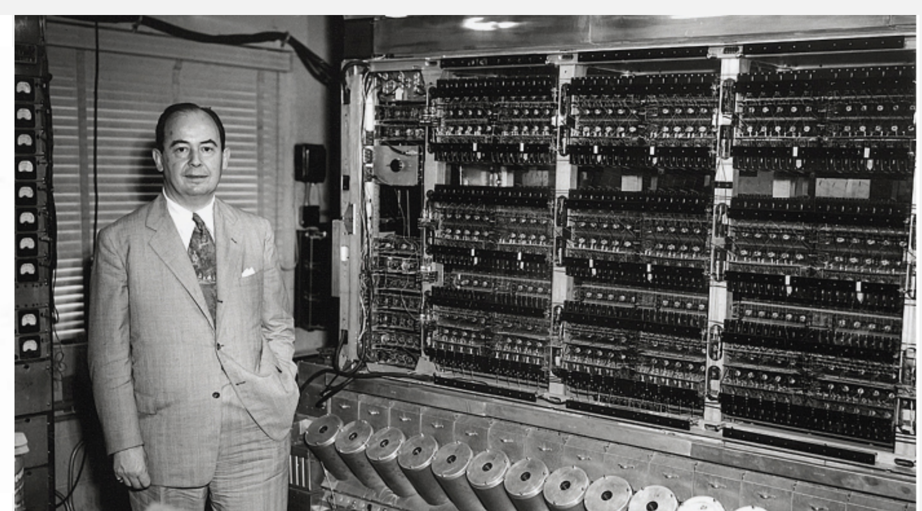
- Divide array into two halves.
- Recursively sort each half.
- Merge two halves.

input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Mergesort overview


First Draft of a Report on the EDVAC

John von Neumann



Abstract in-place merge demo

Goal. Given two sorted subarrays `a[lo]` to `a[mid]` and `a[mid+1]` to `a[hi]`, replace with sorted subarray `a[lo]` to `a[hi]`.

	lo				mid	mid+1				hi
a[]	E	E	G	M	R	A	C	E	R	T

sorted

sorted

Mergesort: Transylvanian–Saxon folk dance

http://www.youtube.com/watch?v=XaqR3G_NVoo

Merging: Java implementation

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }
}
```

copy

merge

	lo			i	mid			j		hi
aux[]	A	G	L	O	R	H	I	M	S	T

						k				
a[]	A	G	H	I	L	M				

Mergesort quiz 1

How many calls does `merge()` make to `less()` in order to merge two sorted subarrays, each of length $n / 2$, into a sorted array of length n?

A. $\sim \frac{1}{4} n$ to $\sim \frac{1}{2} n$

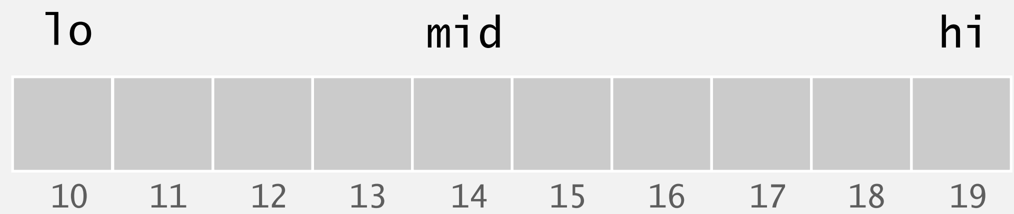
B. $\sim \frac{1}{2} n$

C. $\sim \frac{1}{2} n$ to $\sim n$

D. $\sim n$

Mergesort: Java implementation

```java
public class Merge
{
   private static void merge(...)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      Comparable[] aux = new Comparable[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```



Mergesort: trace

a[]

lo, hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 0, 0, 1)`	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 2, 2, 3)`	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 0, 1, 3)`	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 4, 4, 5)`	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 6, 6, 7)`	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 4, 5, 7)`	E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
`merge(a, aux, 0, 3, 7)`	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
`merge(a, aux, 8, 8, 9)`	E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
`merge(a, aux, 10, 10, 11)`	E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
`merge(a, aux, 8, 9, 11)`	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
`merge(a, aux, 12, 12, 13)`	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
`merge(a, aux, 14, 14, 15)`	E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
`merge(a, aux, 12, 13, 15)`	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
`merge(a, aux, 8, 11, 15)`	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
`merge(a, aux, 0, 7, 15)`	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

result after recursive call

Mergesort quiz 2

Which of the following subarray lengths will occur when running mergesort on an array of length 12?

A. $\{\ 1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12\ \}$

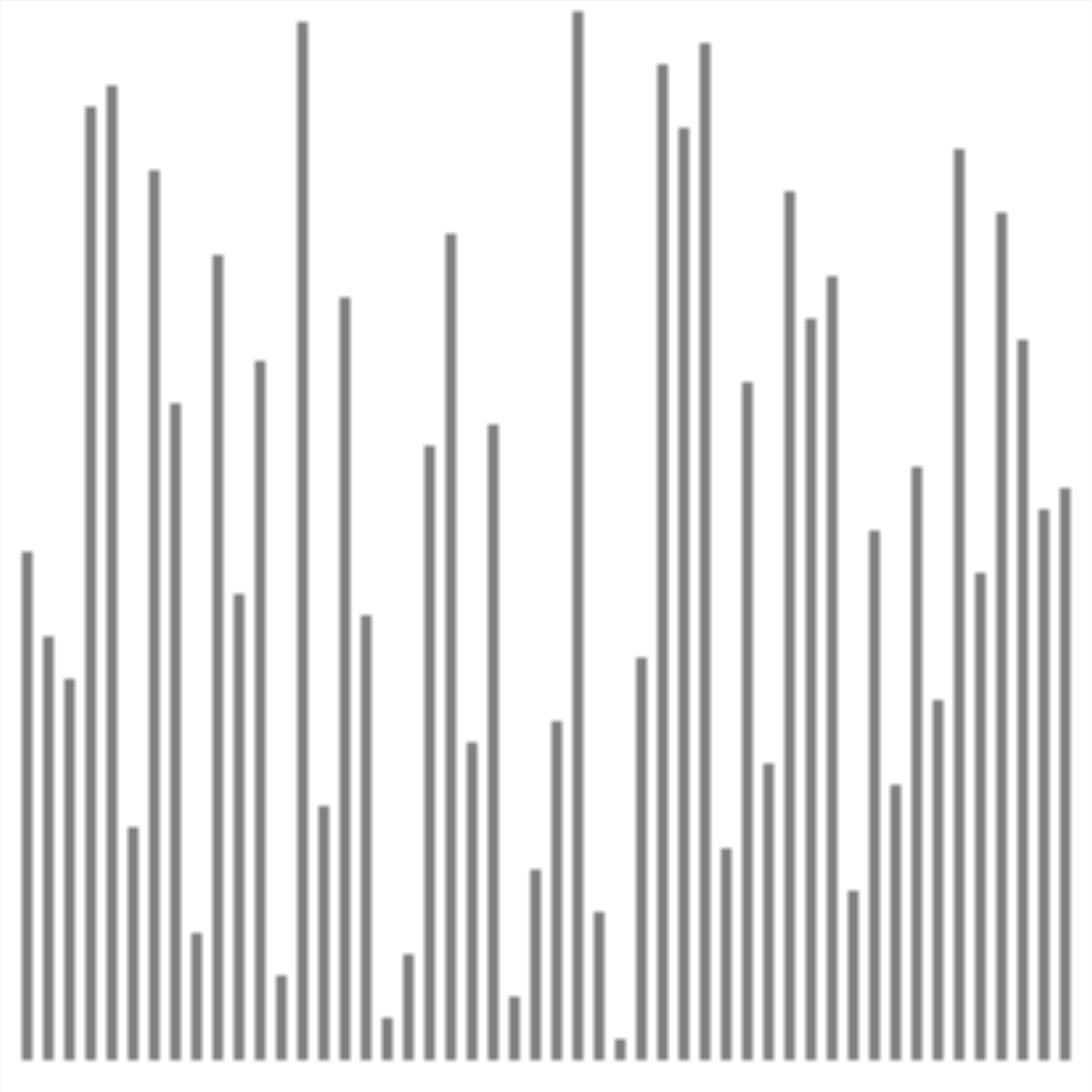
B. $\{\ 1,\ 2,\ 3,\ 6,\ 12\ \}$

C. $\{\ 1,\ 2,\ 4,\ 8,\ 12\ \}$

D. $\{\ 1,\ 3,\ 6,\ 9,\ 12\ \}$

Mergesort: animation

50 random items

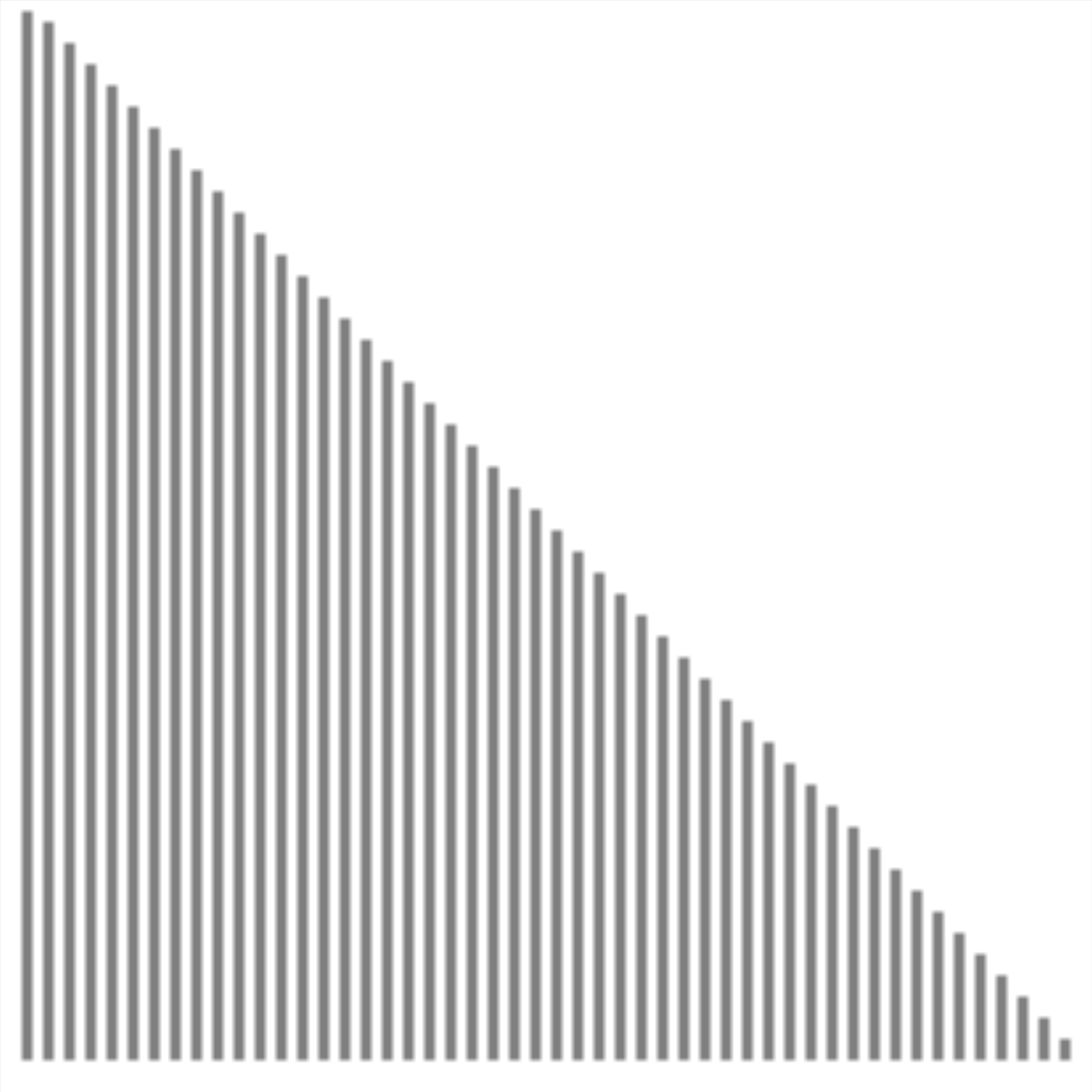


http://www.sorting-algorithms.com/merge-sort

▲ algorithm position
in order
current subarray
not in order

Mergesort: animation

50 reverse-sorted items



http://www.sorting-algorithms.com/merge-sort

▲ algorithm position

in order

current subarray

not in order

Mergesort: empirical analysis

Running time estimates:

- Laptop executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (n^2)			mergesort ($n \log n$)		
computer	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min
super	instant	1 second	1 week	instant	instant	instant

Bottom line. Good algorithms are better than supercomputers.

Mergesort analysis: number of compares

Proposition. Mergesort uses $\leq n \lg n$ compares to sort an array of length n.

Pf sketch. The number of compares $C(n)$ to mergesort an array of length n satisfies the recurrence:

$$C(n) \leq \underbrace{C(\lceil n/2 \rceil)}_{\text{left half}} + \underbrace{C(\lfloor n/2 \rfloor)}_{\text{right half}} + \underbrace{n-1}_{\text{merge}} \quad \text{for } n > 1, \text{ with } C(1) = 0.$$

We solve this simpler recurrence, and assume n is a power of 2:

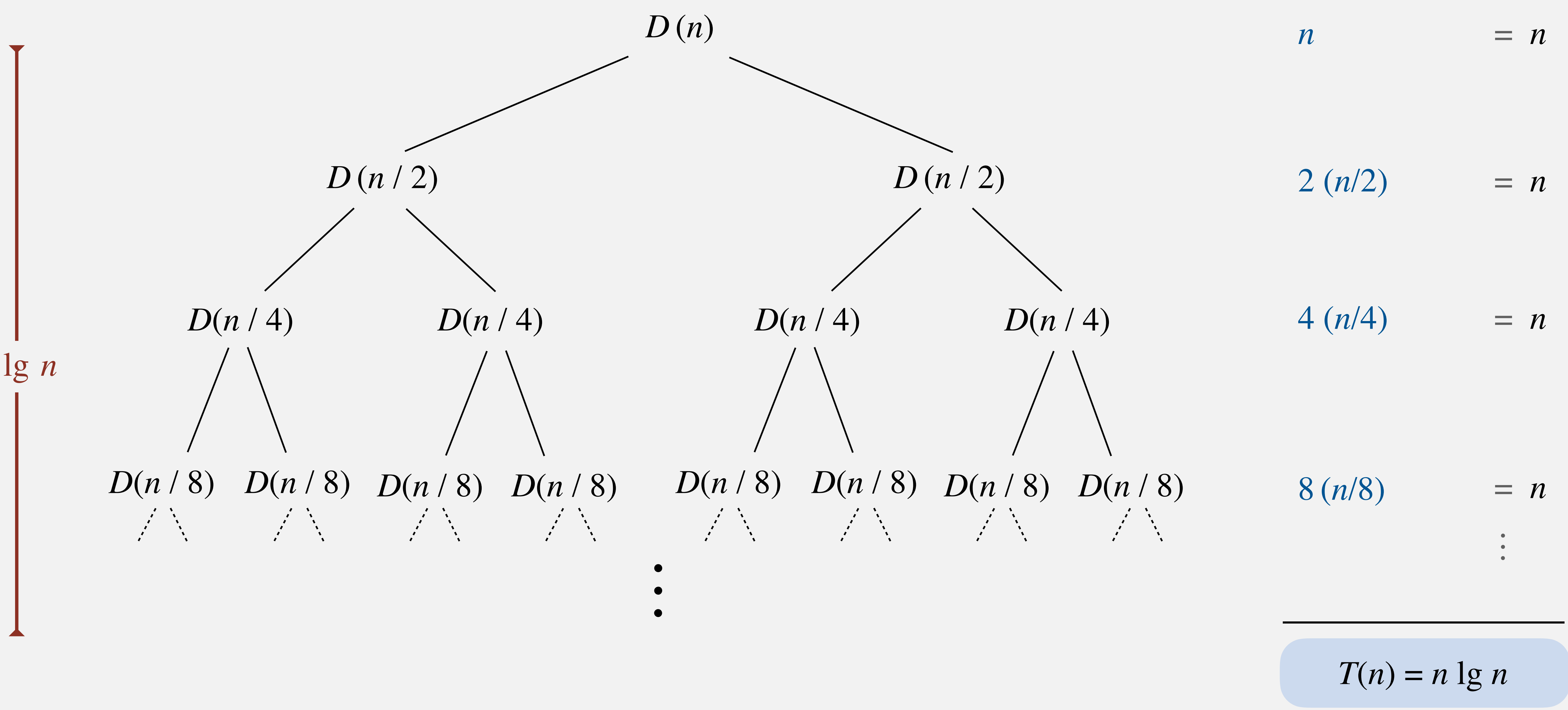
result holds for all n
(analysis cleaner in this case)

$$D(n) = 2\,D(n/2) + n, \text{ for } n > 1, \text{ with } D(1) = 0.$$

Divide-and-conquer recurrence

Proposition. If $D(n)$ satisfies $D(n) = 2\,D(n/2) + n$ for $n > 1$, with $D(1) = 0$, then $D(n) = n \lg n$.

Pf by picture. [assuming n is a power of 2]



Mergesort analysis: number of array accesses

Proposition. Mergesort uses $\le 6\, n \lg n$ array accesses to sort an array of length n.

Pf sketch. The number of array accesses $A(n)$ satisfies the recurrence:

$$A(n) \le A(\lceil n/2 \rceil) + A(\lfloor n/2 \rfloor) + 6n \quad \text{for } n > 1, \text{ with } A(1) = 0.$$

Key point. Any algorithm with the following structure takes $n \log n$ time:

```
public static void f(int n)
{
    if (n == 0) return;
    f(n/2);            ← solve two problems
    f(n/2);            ← of half the size
    linear(n);         ← do a linear amount of work
}
```

Notable examples. FFT, hidden-line removal, Kendall-tau distance, …

Mergesort analysis: memory

Proposition. Mergesort uses extra space proportional to n.

Pf. The array `aux[]` needs to be of length n for the last merge.

two sorted subarrays

A	C	D	G	H	I	M	N	U	V

B	E	F	J	O	P	Q	R	S	T

merged result

A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V

Def. A sorting algorithm is in-place if it uses $\le c \log n$ extra memory.

Ex. Insertion sort and selection sort.

Challenge 1 (not hard). Use `aux[]` array of length $\sim \frac{1}{2} n$ instead of n.

Challenge 2 (very hard). In-place merge. [Kronrod 1969]

Mergesort quiz 3

Is our implementation of mergesort stable?

A. Yes.

B. No, but it can be easily modified to be stable.

C. No, mergesort is inherently unstable.

D. *I don't remember what stability means.*

a sorting algorithm is stable if it preserves the relative order of equal keys

input	C	A_1	B	A_2	A_3
sorted	A_3	A_1	A_2	B	C

not stable

Stability: mergesort

Proposition. Mergesort is stable.

```
public class Merge
{
   private static void merge(...)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {  /* as before */  }
}
```

Pf. Suffices to verify that merge operation is stable.

Stability: mergesort

Proposition. Merge operation is stable.

```
private static void merge(...)
{
   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }
}
```

0	1	2	3	4
A_1	A_2	A_3	B	D

5	6	7	8	9	10
A_4	A_5	C	E	F	G

Pf. Takes from left subarray if equal keys.

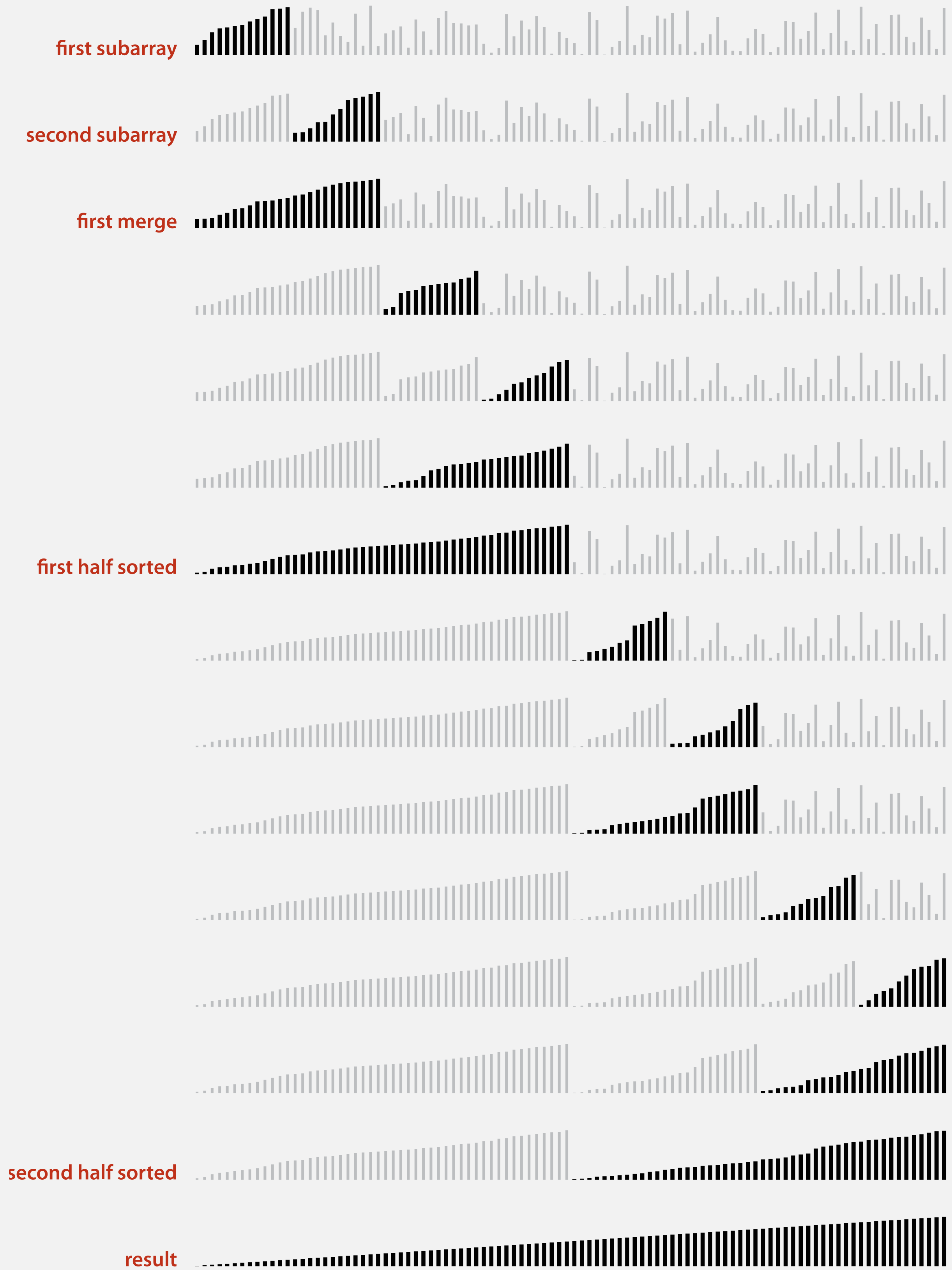
Mergesort: practical improvements

Use insertion sort for small subarrays.

- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.

```
private static void sort(...)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

Mergesort with cutoff to insertion sort: visualization



Mergesort: practical improvements

Stop if already sorted.

- Is largest item in first half ≤ smallest item in second half?
- Helps for partially ordered arrays.

A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V

A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V

```
private static void sort(...)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);

   if (!less(a[mid+1], a[mid])) return;

   merge(a, aux, lo, mid, hi);
}
```

Mergesort: practical improvements

Eliminate the copy to the auxiliary array. Save time (but not space) by switching the role of the input and auxiliary array in each recursive call.

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)          aux[k] = a[j++];
      else if (j > hi)           aux[k] = a[i++];
      else if (less(a[j], a[i])) aux[k] = a[j++];
      else                       aux[k] = a[i++];
   }
}

private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (aux, a, lo, mid);
   sort (aux, a, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

merge from a[] to aux[]

assumes aux[] is initialize to a[] once, before recursive calls

switch roles of aux[] and a[]

Java 6 system sort

Basic algorithm in `Arrays.sort()` for sorting objects = mergesort.

- Cutoff to insertion sort = 7.
- Stop-if-already-sorted test.
- Eliminate-the-copy-to-the-auxiliary-array trick.

http://hg.openjdk.java.net/jdk6/jdk6/jdk/file/tip/src/share/classes/java/util/Arrays.java

http://algs4.cs.princeton.edu

2.2 Mergesort

- mergesort
- bottom-up mergesort
- sorting complexity
- divide-and-conquer

Bottom-up mergesort

Basic plan.

- Pass through array, merging subarrays of size 1.
- Repeat for subarrays of size 2, 4, 8, ...

a[i]

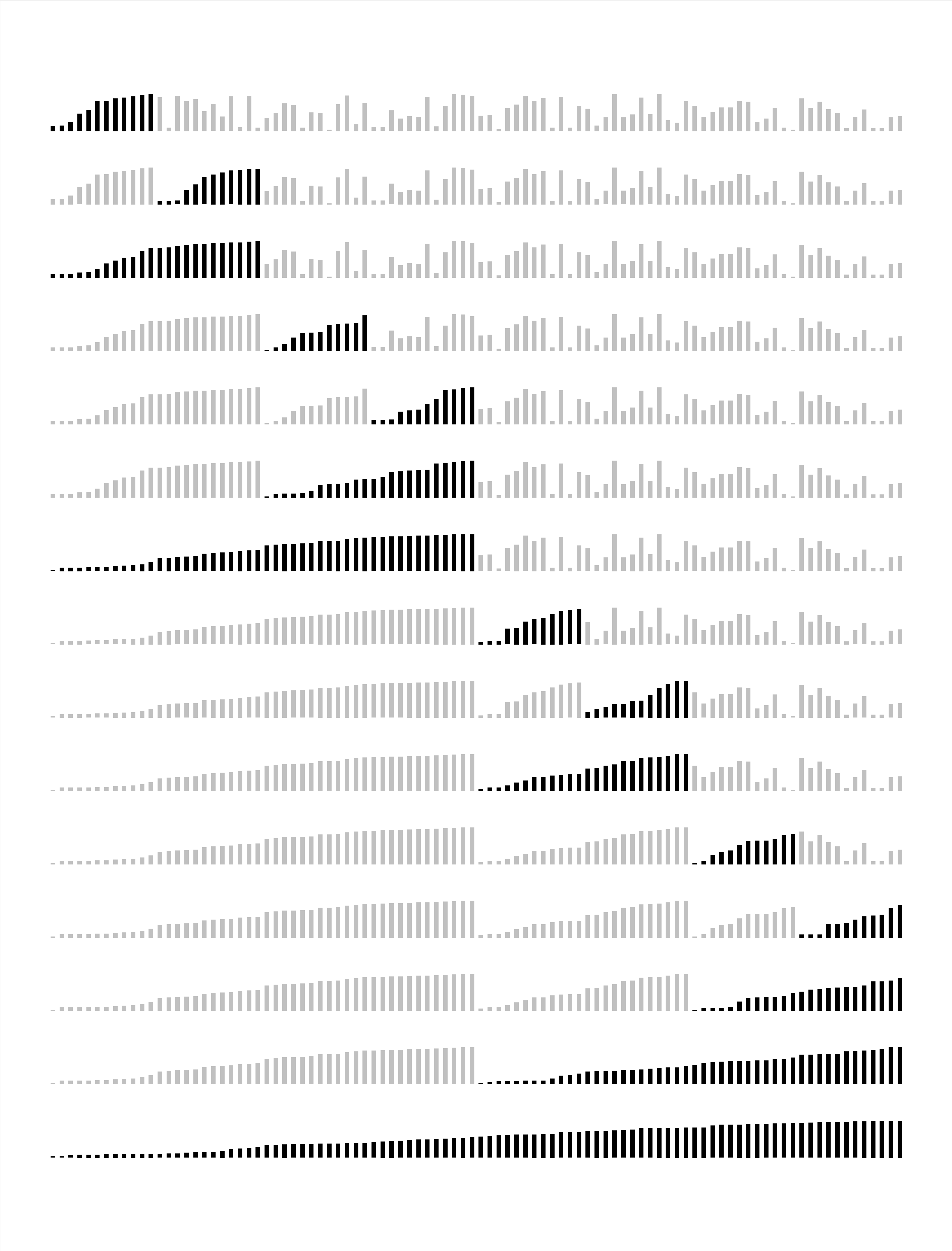
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sz = 1																
`merge(a, aux, 0, 0, 1)`	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 2, 2, 3)`	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 4, 4, 5)`	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 6, 6, 7)`	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 8, 8, 9)`	E	M	G	R	E	S	O	R	E	T	X	A	M	P	L	E
`merge(a, aux, 10, 10, 11)`	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
`merge(a, aux, 12, 12, 13)`	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
`merge(a, aux, 14, 14, 15)`	E	M	G	R	E	S	O	R	E	T	A	X	M	P	E	L
sz = 2																
`merge(a, aux, 0, 1, 3)`	E	G	M	R	E	S	O	R	E	T	A	X	M	P	E	L
`merge(a, aux, 4, 5, 7)`	E	G	M	R	E	O	R	S	E	T	A	X	M	P	E	L
`merge(a, aux, 8, 9, 11)`	E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L
`merge(a, aux, 12, 13, 15)`	E	G	M	R	E	O	R	S	A	E	T	X	E	L	M	P
sz = 4																
`merge(a, aux, 0, 3, 7)`	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
`merge(a, aux, 8, 11, 15)`	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
sz = 8																
`merge(a, aux, 0, 7, 15)`	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Bottom-up mergesort: Java implementation

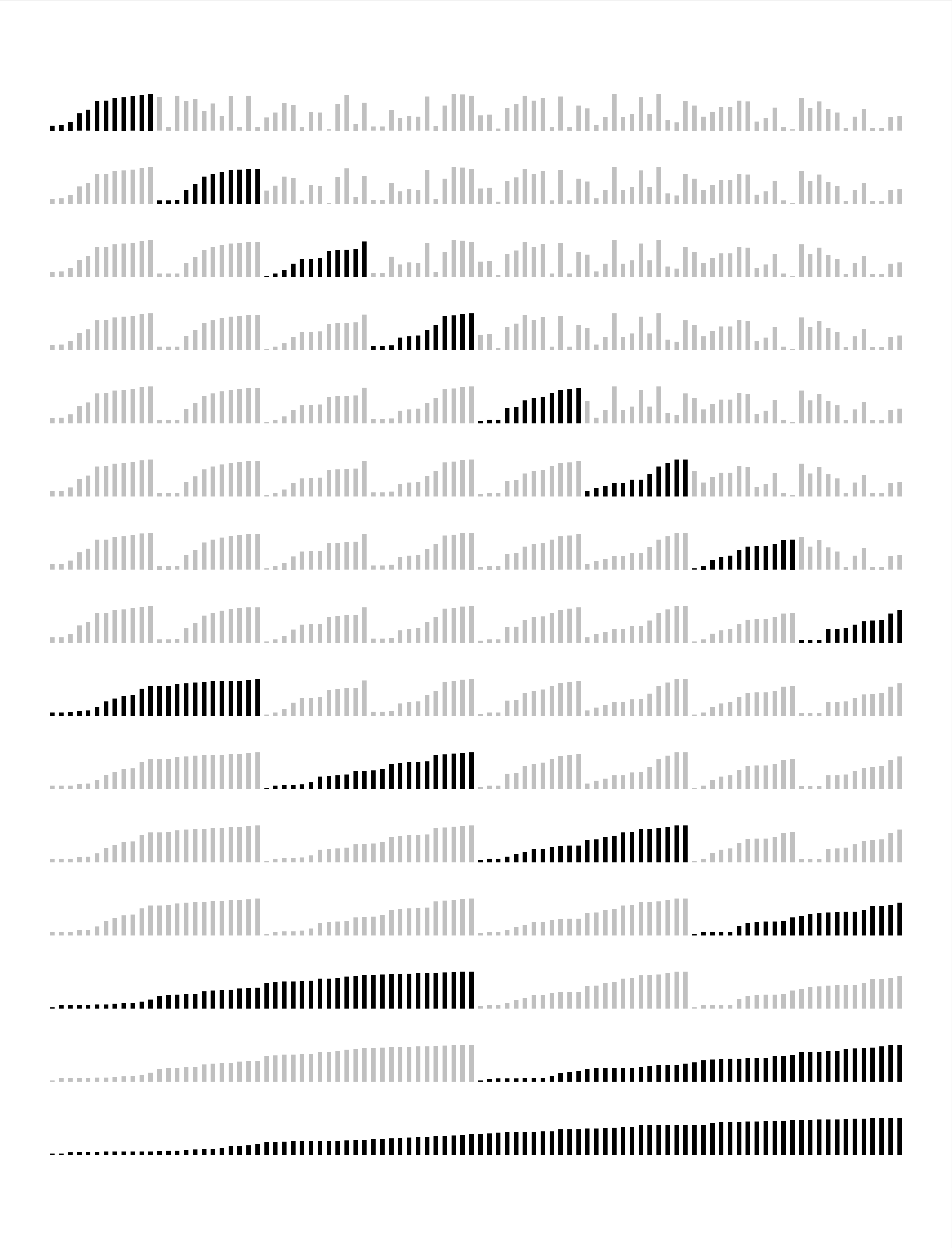
```
public class MergeBU
{
   private static void merge(...)
   {  /* as before */  }

   public static void sort(Comparable[] a)
   {
      int n = a.length;
      Comparable[] aux = new Comparable[n];
      for (int sz = 1; sz < n; sz = sz+sz)
         for (int lo = 0; lo < n-sz; lo += sz+sz)
            merge(a, aux, lo, lo+sz-1, Math.min(lo+sz+sz-1, n-1));
   }
}
```

Bottom line. Simple and non-recursive version of mergesort.

Mergesort: visualizations



top-down mergesort (cutoff = 12)



bottom-up mergesort (cutoff = 12)

Mergesort quiz 4

Which is faster in practice: top-down mergesort or bottom-up mergesort? You may assume n is a power of 2.

A. Top-down (recursive) mergesort.

B. Bottom-up (non-recursive) mergesort.

C. No observable difference.

D. *I don't know.*

Natural mergesort

Idea. Exploit pre-existing order by identifying naturally occurring runs.

input

1	5	10	16	3	4	23	9	13	2	7	8	12	14

first run

1	5	10	16	3	4	23	9	13	2	7	8	12	14

second run

1	5	10	16	3	4	23	9	13	2	7	8	12	14

merge two runs

1	3	4	5	10	16	23	9	13	2	7	8	12	14

Tradeoff. Fewer passes vs. extra compares per pass to identify runs.

Timsort

- Natural mergesort.
- Use binary insertion sort to make initial runs (if needed).
- A few more clever optimizations.

```
Intro
-----
This describes an adaptive, stable, natural mergesort, modestly called
timsort (hey, I earned it <wink>).  It has supernatural performance on many
kinds of partially ordered arrays (less than lg(n!) comparisons needed, and
as few as n-1), yet as fast as Python's previous highly tuned samplesort
hybrid on random arrays.

In a nutshell, the main routine marches over the array once, left to right,
alternately identifying the next run, then merging it into the previous
runs "intelligently".  Everything else is complication for speed, and some
hard-won measure of memory efficiency.
...
```

Tim Peters

Consequence. Linear time on many arrays with pre-existing order.

Now widely used. Python, Java 7–9, GNU Octave, Android, ….

http://hg.openjdk.java.net/jdk7/jdk7/jdk/file/tip/src/share/classes/java/util/Arrays.java

Timsort bug

Envisage: Engineering Virtualized Services

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Proving that Android's, Java's and Python's sorting algorithm is broken (and showing how to fix it)

February 24, 2015 Envisage Written by Stijn de Gouw. $s

Tim Peters developed the Timsort hybrid sorting algorithm in 2002. It is a clever combination of ideas from merge sort and insertion sort, and designed to perform well on real world data. TimSort was first developed for Python, but later ported to Java (where it appears as java.util.Collections.sort and java.util.Arrays.sort) by Joshua Bloch (the designer of Java Collections who also pointed out that most binary search algorithms were broken). TimSort is today used as the default sorting algorithm for Android SDK, Sun's JDK and OpenJDK. Given the popularity of these platforms this means that the number of computers, cloud services and mobile phones that use TimSort for sorting is well into the billions.

http://envisage-project.eu/proving-android-java-and-python-sorting-algorithm-is-broken-and-how-to-fix-it

Sorting summary

	inplace?	stable?	best	average	worst	remarks
selection	✔		$\frac{1}{2}\, n^2$	$\frac{1}{2}\, n^2$	$\frac{1}{2}\, n^2$	n exchanges
insertion	✔	✔	n	$\frac{1}{4}\, n^2$	$\frac{1}{2}\, n^2$	use for small n or partially ordered
shell	✔		$n \log_3 n$	?	$c\, n^{3/2}$	tight code; subquadratic
merge		✔	$\frac{1}{2}\, n \lg n$	$n \lg n$	$n \lg n$	$n \log n$ guarantee; stable
timsort		✔	n	$n \lg n$	$n \lg n$	improves mergesort when pre-existing order
?	✔	✔	n	$n \lg n$	$n \lg n$	holy sorting grail

http://algs4.cs.princeton.edu

2.2 Mergesort

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Complexity of sorting

Computational complexity. Framework to study efficiency of algorithms for solving a particular problem X.

Model of computation. Allowable operations.
Cost model. Operation counts.
Upper bound. Cost guarantee provided by some algorithm for X.
Lower bound. Proven limit on cost guarantee of all algorithms for X.
Optimal algorithm. Algorithm with best possible cost guarantee for X.

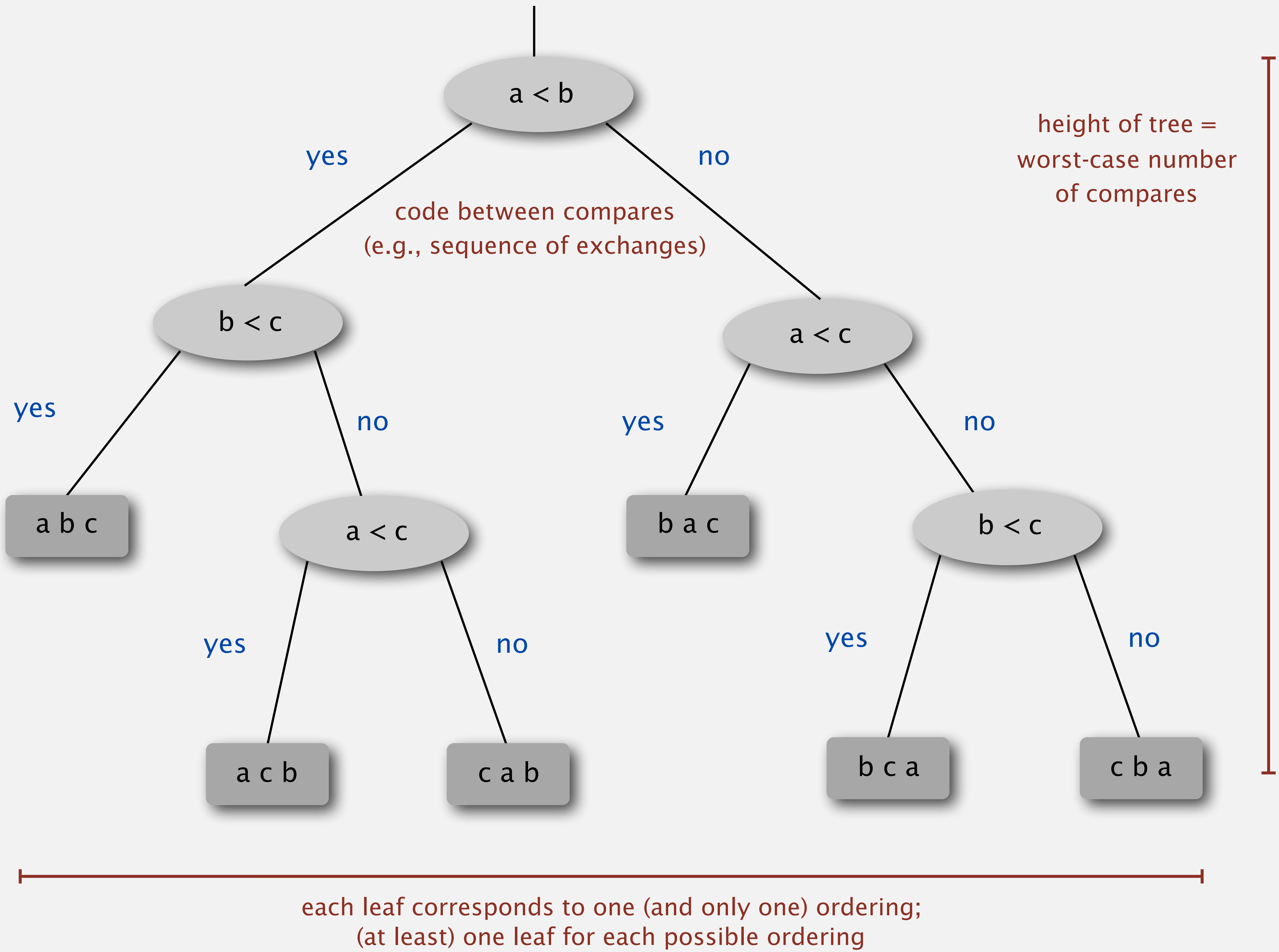
lower bound ~ upper bound

model of computation	*decision tree*
cost model	*# compares*
upper bound	$\sim n \lg n$ *from mergesort*
lower bound	?
optimal algorithm	?

can access information only through compares (e.g., Java `Comparable` framework)

complexity of sorting

Decision tree (for 3 distinct keys a, b, and c)

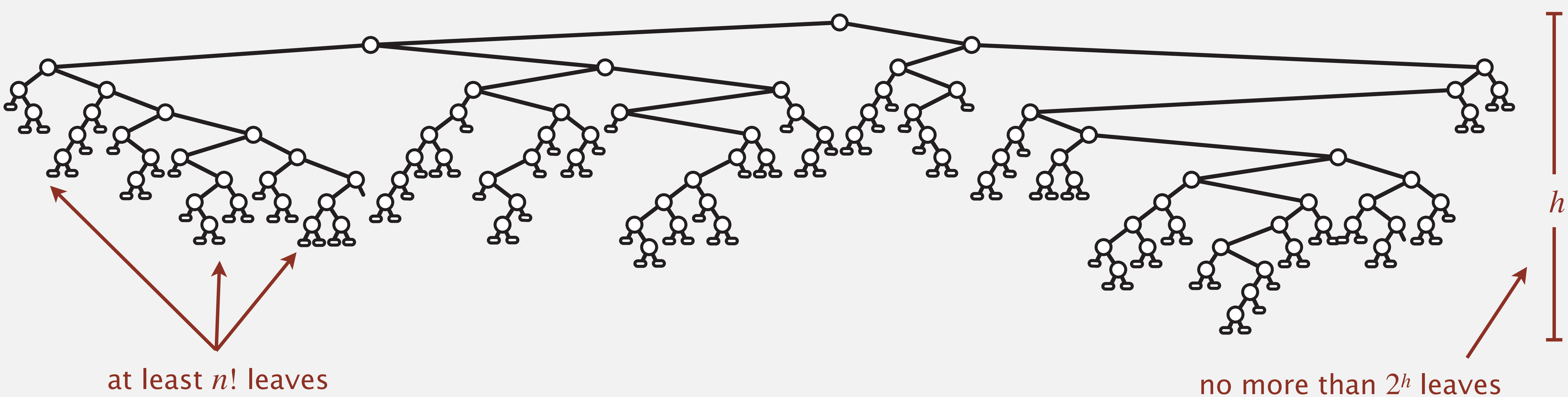


Compare-based lower bound for sorting

Proposition. Any compare-based sorting algorithm must make at least $\lg(n!) \sim n \lg n$ compares in the worst-case.

Pf.

- Assume array consists of n distinct values a_1 through a_n.
- Worst case dictated by height h of decision tree.
- Binary tree of height h has at most 2^h leaves.
- $n!$ different orderings $\Rightarrow$ at least $n!$ leaves.



Compare-based lower bound for sorting

Proposition. Any compare-based sorting algorithm must make at least $\lg(n!) \sim n \lg n$ compares in the worst-case.

Pf.

- Assume array consists of n distinct values a_1 through a_n.
- Worst case dictated by **height** h of decision tree.
- Binary tree of height h has at most 2^h leaves.
- $n!$ different orderings $\Rightarrow$ at least $n!$ leaves.

$$2^h \geq \#\,\text{leaves} \geq n!$$

$$\Rightarrow \; h \geq \lg(n!) \sim n \lg n$$

Stirling's formula

Complexity of sorting

Model of computation. Allowable operations.

Cost model. Operation count(s).

Upper bound. Cost guarantee provided by some algorithm for X.

Lower bound. Proven limit on cost guarantee of all algorithms for X.

Optimal algorithm. Algorithm with best possible cost guarantee for X.

model of computation	*decision tree*
cost model	*# compares*
upper bound	$\boxed{\sim n \lg n}$
lower bound	$\boxed{\sim n \lg n}$
optimal algorithm	$\boxed{\textit{mergesort}}$

complexity of sorting

First goal of algorithm design: optimal algorithms.

Complexity results in context

Compares? Mergesort is optimal with respect to number compares.

Space? Mergesort is not optimal with respect to space usage.

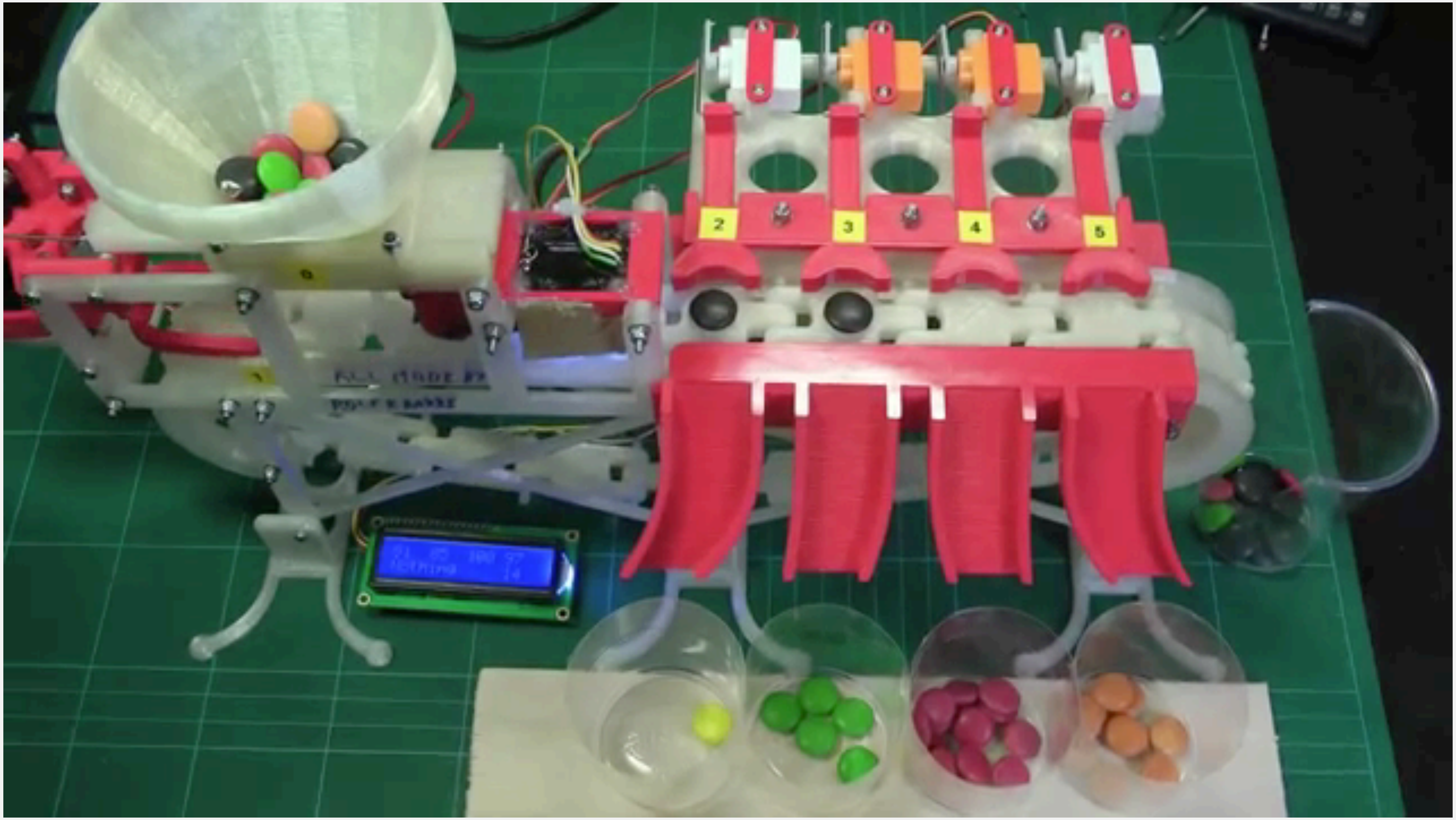
Lessons. Use theory as a guide.

Ex. Design sorting algorithm that guarantees $\sim \frac{1}{2}\, n \lg n$ compares?

Ex. Design sorting algorithm that is both time- and space-optimal?

Commercial break

Q. Why doesn't this Skittles machine violate the sorting lower bound?



https://www.youtube.com/watch?v=tSEHDBSynVo

Complexity results in context (continued)

Lower bound may not hold if the algorithm can take advantage of:

- The initial order of the input array.
  Ex: insertion sort requires only a linear number of compares on partially sorted arrays.
- The distribution of key values.
  Ex: 3-way quicksort requires only a linear number of compares on arrays with a constant number of distinct keys. [stay tuned]
- The representation of the keys.
  Ex: radix sorts require no key compares — they access the data via character/digit compares. [stay tuned]

Commonly used notations in the theory of algorithms

notation	provides	example	shorthand for
Tilde	leading term	$\sim \frac{1}{2} n^2$	$\frac{1}{2} n^2$ $\frac{1}{2} n^2 + 22 n \log n + 3 n$
Big Theta	order of growth	$\Theta(n^2)$	$\frac{1}{2} n^2$ $10 n^2$ $5 n^2 + 22 n \log n + 3 n$
Big O	upper bound	$O(n^2)$	$10 n^2$ $100 n$ $22 n \log n + 3 n$
Big Omega	lower bound	$\Omega(n^2)$	$\frac{1}{2} n^2$ n^5 $n^3 + 22 n \log n + 3 n$

Sorting lower bound

Interviewer. Give a formal description of the sorting lower bound for sorting an array of n elements.

Cornell student. I use `Arrays.sort()`.

Yale student. Any sorting algorithm takes at least $n \log n$ time.

Harvard student. Any compare-based sorting algorithm must make at least $O(n \log n)$ compares.

Princeton student. To sort an arbitrary array of n items, any compare-based sorting algorithm must make $\Omega(n \log n)$ compares in the worst case.

http://algs4.cs.princeton.edu

2.2 Mergesort

- mergesort
- bottom-up mergesort
- sorting complexity
- *divide-and-conquer*

Sorting a linked list

Problem. Given a singly linked list, rearrange its nodes in sorter order.

Version 1. Linearithmic time, linear extra space.

Version 2. Linearithmic time, logarithmic (or constant) extra space.