Non-Rigid Surface Correspondence

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Goal

Find maps between surfaces

- Non-rigid
- Bijective
- Smooth
- Shape preserving
- Automatic
- Efficient computation
- Provide metric
- Semantic alignment
Motivating Applications

Finding corresponding points on surfaces enables:

- Surface comparison
- Collection analysis
- Property transfer
- Morphing
- etc.

[Praun et al.]
Problem 1

Find a sparse set of feature correspondences
Problem 2

Compute a dense map from a sparse set of feature correspondences

Least Squares Conformal Map (preserve angles as best as possible)

Zeng et al., 2008
Outline

Introduction

Some surface mapping algorithms
  - Feature correspondence search
  - High-dimensional embedding
  - Möbius transformations
  - Blended maps

Example Application

Conclusion

Future work
Outline

Introduction

Some surface mapping algorithms

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Feature Correspondence Search

For each coarse set of feature correspondences:

- Measure the deformation required to align them
  - … maybe by solving problem 2
- Remember the one with least deformation
Feature Correspondence Search

Measures of distortion:

- Differences in geodesic distances
- Differences in conformal factors (angles)
- etc.

[Branch and bound Priority-driven search etc.]

Least squares conformal map aligning corresponding feature points

[Feature points]

[Zeng et al., 2008]
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    ➢ High-dimensional embedding
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High-Dimensional Embedding

Find nearest neighbors after spectral embedding

Eigenfunctions of the Laplacian
[Lombaert et al. 2011]
High-Dimensional Embedding

Find nearest neighbors after spectral embedding

Eigenfunctions of the Laplacian

[Lombaert et al. 2011]
High-Dimensional Embedding

Find nearest neighbors after heat kernel embedding implied by a single point correspondence

Heat Kernel Map
[Ovsjanikov et al. 2010]
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Möbius Transformations

It would be nice to search a low-dimensional space of transformations to align non-rigid surfaces …

Scan A  
Scan B  
Best Alignment

RANSAC  
Hough transform  
Geometric hashing  
etc.
Key Observation

The Möbius group provides a low-dimensional space to search efficiently for the “best” conformal map between genus zero surfaces.
Möbius Transformations I

Möbius transformations are a group of functions on the extended complex plane that represent bijective, conformal maps.
Möbius Transformations II

Möbius transformations are simple rational functions:

\[ f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \quad a, b, c, d \in \mathbb{C} \]

They have only six degrees of freedom (they can be computed analytically from three point correspondences)

\[ f(z_i) = y_i, i = 1, 2, 3 \]
Möbius Transformations III

Therefore, any three point correspondences define a bijective, conformal map from the extended complex plane onto itself.
Möbius Transformations IV

Since every genus zero surface can be mapped conformally onto the extended complex plane (Riemann sphere), …
Any three point correspondences define a bijective, conformal map between genus zero surfaces.

Surface A

Surface B

Uniformization

Bijective Conformal

Uniformization⁻¹

Extended complex plane

Mobius Transformation
We can search for the “lowest distortion” bijective, conformal map between genus zero surfaces using algorithms that sample triplets of correspondences (e.g., RANSAC, Hough transform, etc.)

Polynomial-time algorithm for non-rigid surface mapping
Surface Mapping Algorithm

Example: RANSAC algorithm

For $i = 1$ to $\sim N^3$

Sample three points $(A_1, A_2, A_3)$ on surface $A$
Sample three points $(B_1, B_2, B_3)$ on surface $B$

Compute conformal map $M: (A_1, A_2, A_3) \rightarrow (B_1, B_2, B_3)$
Remember $M$ if distortion is smallest
Surface Mapping Algorithm

Example: RANSAC algorithm

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Measure distortion by relative change of area
(deviation from isometry)
Surface Mapping Algorithm

Example: RANSAC algorithm

For $i = 1$ to $\approx N^3$

- Sample three points $(A1, A2, A3)$ on surface $A$
- Sample three points $(B1, B2, B3)$ on surface $B$
- Compute conformal map $M: (A1, A2, A3) \rightarrow (B1, B2, B3)$
- Remember $M$ if distortion is smallest

Measure distortion by relative change of area
(deviation from isometry)
Surface Mapping Algorithm

RANSAC algorithm properties:

- Non-rigid
- Bijective
- Smooth
- Shape preserving
- Automatic
- Efficient computation
- Provides metric
- Semantic alignment?
Experimental Results

Data:
- 51 pairs of meshes representing animals from TOSCA and SHREC Watertight data sets

Methodology:
- Predict surface maps
- Compare to ground truth semantic correspondences
Experimental Results

Evaluation:

1. For every point with a ground truth correspondence, measure geodesic distance between predicted correspondence and ground truth correspondence.

2. Plot fraction of points within geodesic error threshold.
Experimental Results

Results:

Fraction of correspondences within distance threshold

Geodesic distance threshold \( \frac{x_1}{\sqrt{\text{area}}} \)
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Blended Maps

For significantly different surfaces, no single conformal map provides low distortion everywhere.
Blended Maps

Idea: blend conformal maps with smooth weights

Blending Weights for $m_1$, $m_2$, and $m_3$

Distortion of the Blended Map
Computing Blended Maps

1. Generate consistent set of maps
2. Find blending weights
3. Blend maps
Computing Blended Maps

1. Generate candidate maps by enumerating triplets of feature correspondences

Set of candidate maps
Computing Blended Maps

2. Select consistent set of low-distortion candidate maps

Set of candidate maps
Computing Blended Maps

2a. Define a matrix $B$ where every entry $(i,j)$ indicates the distortion of $m_i$ and $m_j$ and their pairwise similarity $S_{i,j}$

$$B_{i,j} = \int_{M_1} c_i(p) c_j(p) S_{i,j}(p) dA(p)$$
2b. Find block of consistent, low-distortion maps using top eigenvector(s) of $\mathbf{B}$

$$E_M(\vec{w}) = \vec{w}^T \mathbf{B} \vec{w}$$

$$\|\vec{w}\|_2 = 1$$
Computing Blended Maps

3. Compute blending weight $c_i(p)$ for every map $m_i$ at every point $p$ based on distortion of $m_i$ at $p$.

Candidate Map

Blending Weight $c_i(p)$
Computing Blended Maps

4. Define image $p'$ of every point $p$ as the weighted geodesic centroid of $m_i(p)$
Experimental Results

Fraction of correspondences within distance threshold

Geodesic distance threshold ($x_1/\sqrt{\text{area}}$)

Blended Maps (Mobius)

RANSAC (Mobius)

Heat Kernel Maps

GMDS
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Application

Automatically quantify the geometric similarity of anatomical surfaces

[Boyer, Lipman, St. Clair, Puente, Patel, Funkhouser, Jernvall, and Daubechies, 2011]
Application

Traditional Procrustes distance:

\[ d(X, Y) = \min_R \left[ \left( \sum_{i=1}^{N} ||R(X_i) - Y_i||^2 \right)^{1/2} \right] \]

\[ X = \{ X_i \} \]

\[ Y = \{ Y_i \} \]
Application

New continuous Procrustes distance:

\[ d(A, B) = \min_{R,M} \left[ \left( \int_A \| R(x) - M(x) \|^2 \, dx \right)^{1/2} \right] \]
Application

Embedding based on new distance
Application

Clustering based on new distance

Species Groups of Galaga Genus
## Application

### Classification based on nearest-neighbors

<table>
<thead>
<tr>
<th>Mandibular Molar</th>
<th># Groups</th>
<th># Objects</th>
<th>New Distance</th>
<th>Human Landmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Genus</td>
<td>24</td>
<td>99</td>
<td>90.9%</td>
<td>91.9%</td>
</tr>
<tr>
<td>Family</td>
<td>17</td>
<td>106</td>
<td>92.5%</td>
<td>94.3%</td>
</tr>
<tr>
<td>Order</td>
<td>5</td>
<td>116</td>
<td>94.8%</td>
<td>95.7%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>First Metatarsal</th>
<th># Groups</th>
<th># Objects</th>
<th>New Distance</th>
<th>Human1 Landmarks</th>
<th>Human2 Landmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Genus</td>
<td>13</td>
<td>59</td>
<td>79.9%</td>
<td>76.3%</td>
<td>88.1%</td>
</tr>
<tr>
<td>Family</td>
<td>9</td>
<td>61</td>
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<td>83.6%</td>
<td>93.4%</td>
</tr>
<tr>
<td>Superfamily</td>
<td>2</td>
<td>61</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Distal Radius</th>
<th># Groups</th>
<th># Objects</th>
<th>New Distance</th>
<th>Human Landmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Genus</td>
<td>4</td>
<td>45</td>
<td>84.4%</td>
<td>77.7%</td>
</tr>
</tbody>
</table>
Application

Propagating correspondences

Microcebus (Mc)
Observer-Placed Landmarks

Lepilemur (Lp)
Propagated Landmarks

Propagation sequence

Path 1
Mc
↓
Lp

Path 2
Mc
↓
Ma
↓
Lp

Path 3
Mc
↓
Ad
↓
Lm
↓
Lp

*Red circle is around Entoconid
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Application
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