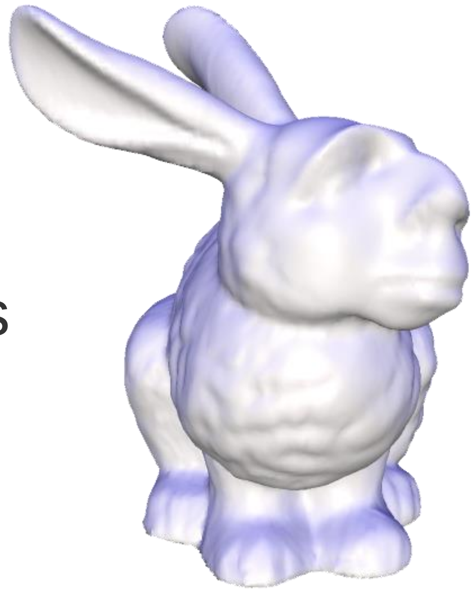
Laplacian Meshes

COS 526 – Fall 2016

Slides from Olga Sorkine and Yaron Lipman

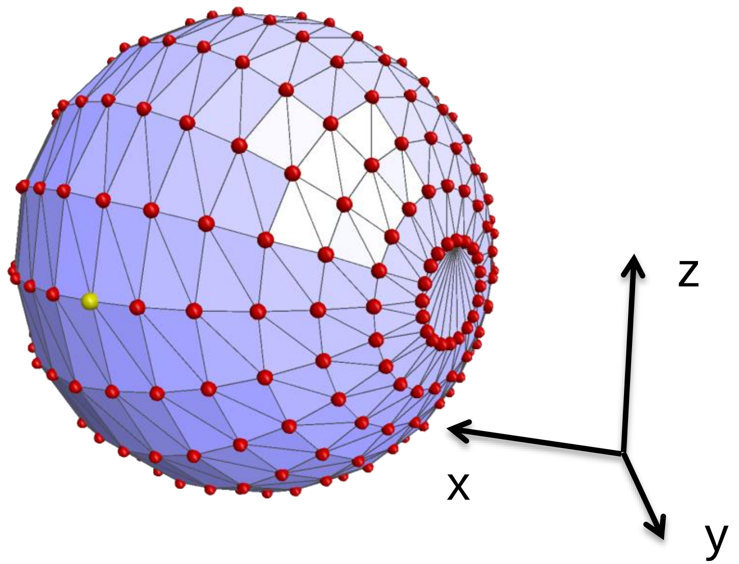
Outline

- Differential surface representation
- Ideas and applications
  - Compact shape representation
  - Mesh editing and manipulation
  - Membrane and flattening
  - Generalizing Fourier basis for surfaces



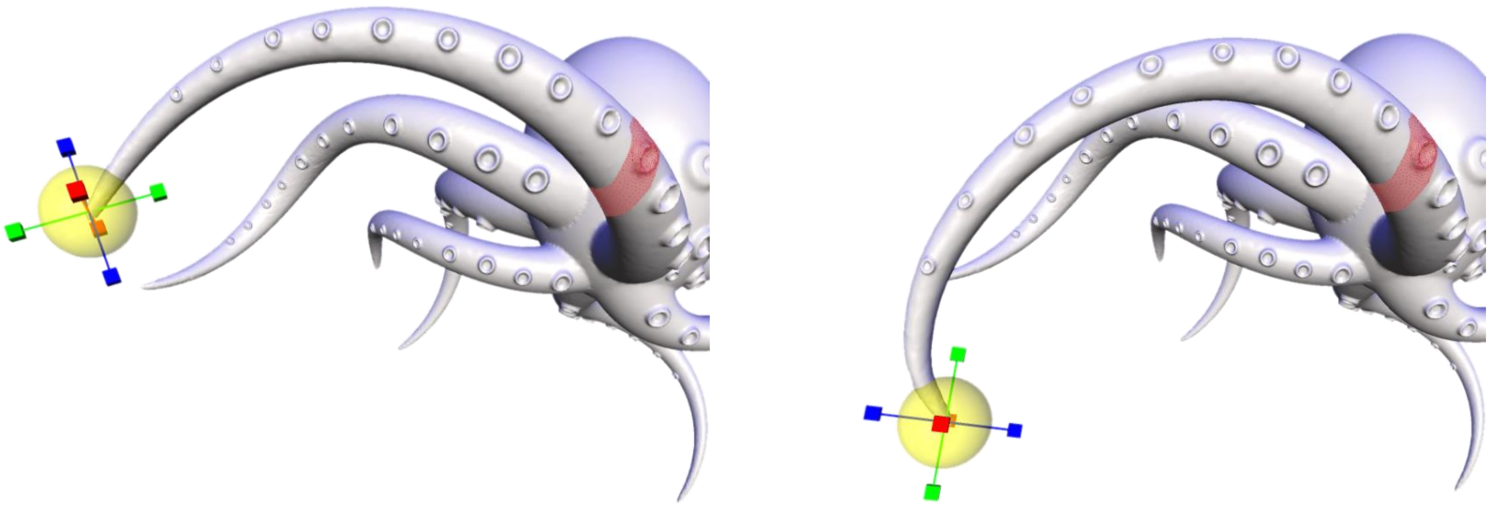
Motivation

- Meshes are great, but:
  - Geometry is represented in a *global* coordinate system
    - Single Cartesian coordinate of a vertex doesn't say much



Laplacian Mesh Editing

- Meshes are difficult to edit



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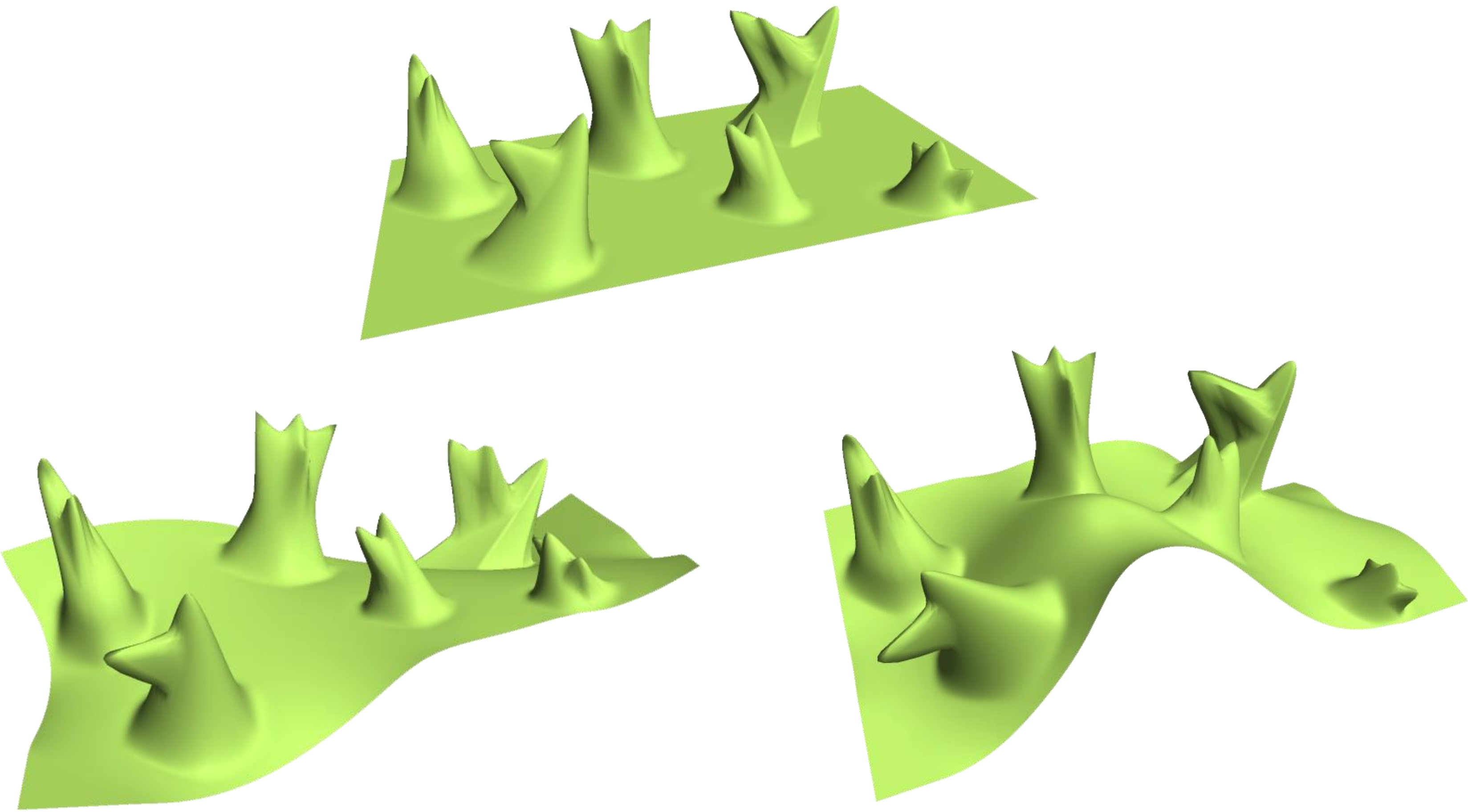

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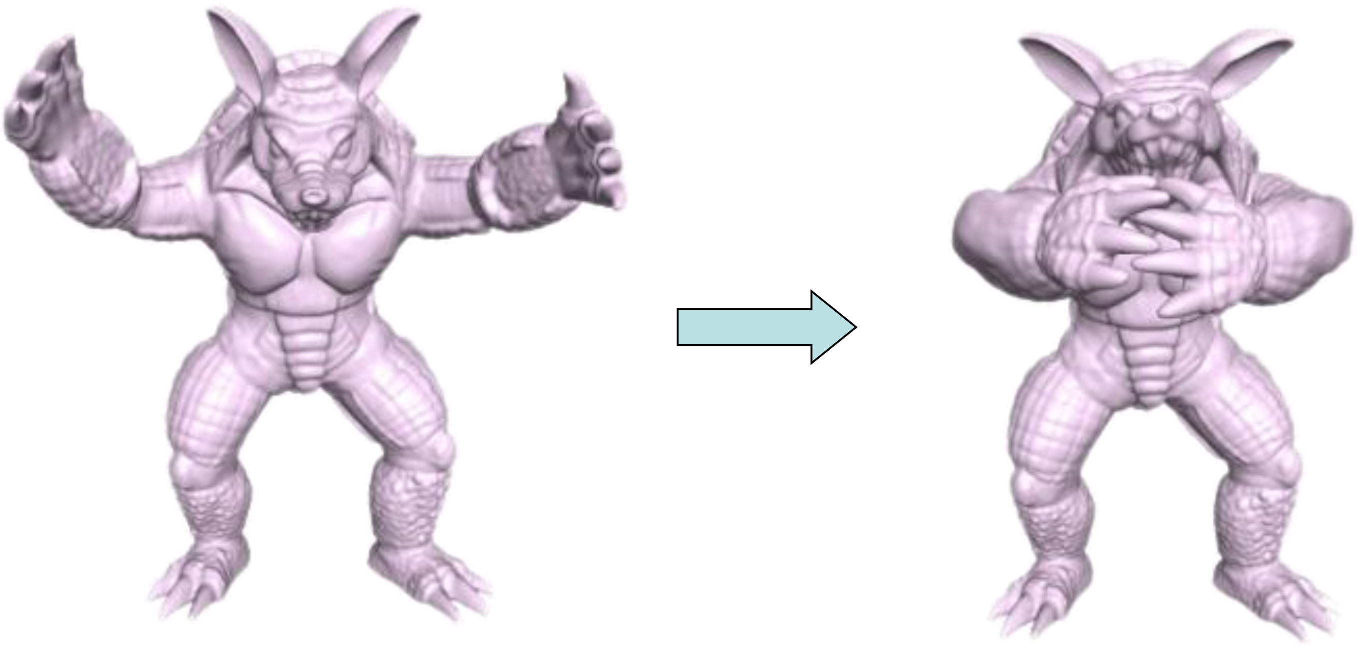


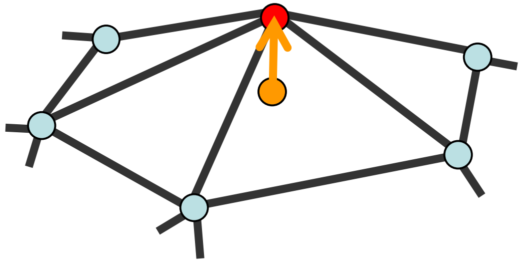
Differential coordinates

- Represent a point ***relative*** to it's neighbors.
- Represent ***local detail*** at each surface point
  - better describe the shape
- Linear transition from global to differential
- Useful for operations on surfaces where surface details are important

Differential coordinates

"Local control for mesh morphing", Alexa 01

- Detail = surface – *smooth*(surface)
- Smoothing = averaging

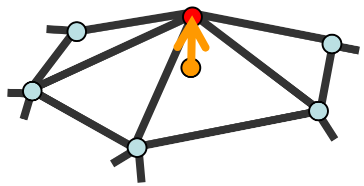


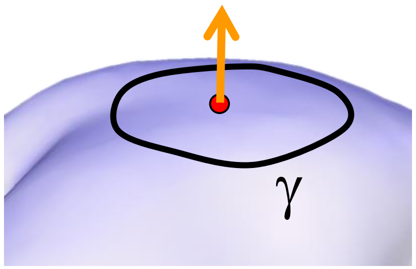
$$\boldsymbol{\delta}_i = \mathbf{v}_i - \frac{1}{d_i} \sum_{j \in N(i)} \mathbf{v}_j$$

$$\boldsymbol{\delta}_i = \sum_{j \in N(i)} \frac{1}{d_i} \left(\mathbf{v}_i - \mathbf{v}_j \right)$$

Connection to the smooth case

- The direction of $\boldsymbol{\delta}_i$ approximates the normal
- The size approximates the mean curvature



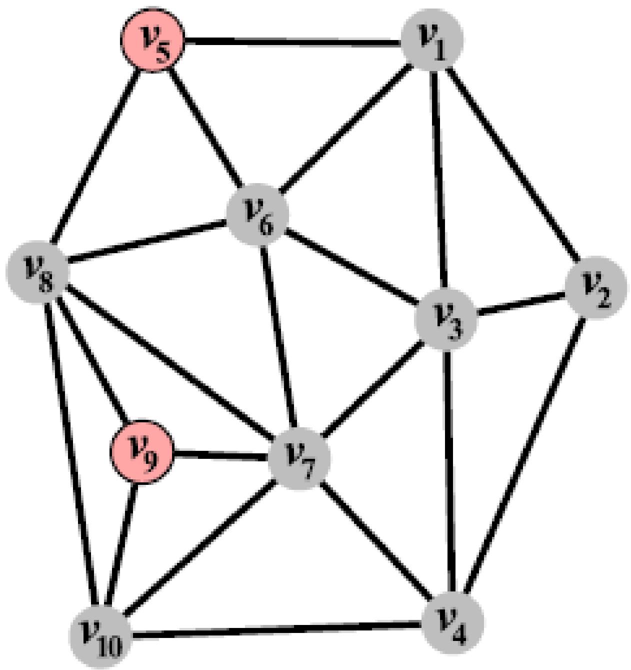


$$\boldsymbol{\delta}_{\mathbf{i}} = \frac{1}{d_i} \sum_{\mathbf{v} \in N(i)} \left(\mathbf{v}_{\mathbf{i}} - \mathbf{v} \right)$$

$$\frac{1}{len(\gamma)} \int_{\mathbf{v} \in \gamma} \left(\mathbf{v}_{\mathbf{i}} - \mathbf{v} \right) ds$$

$$\lim_{len(\gamma) \to 0} \frac{1}{len(\gamma)} \int_{\mathbf{v} \in \gamma} \left(\mathbf{v}_{\mathbf{i}} - \mathbf{v} \right) ds \; = \; H(\mathbf{v}_{\mathbf{i}}) \mathbf{n}_{\mathbf{i}}$$

Laplacian matrix



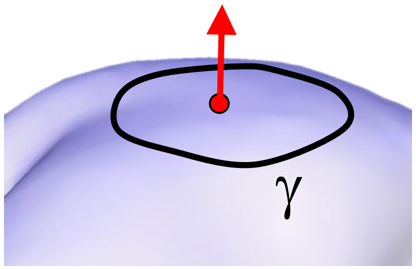
The mesh

$$\begin{bmatrix} 4 & -1 & -1 & & -1 & -1 & & & & \\ -1 & 3 & -1 & -1 & & & & & & \\ -1 & -1 & 5 & -1 & & -1 & -1 & & & \\ & -1 & -1 & 4 & & & -1 & & & -1 \\ -1 & & & & 3 & -1 & & -1 & & \\ -1 & & -1 & & & 4 & -1 & -1 & & \\ & & -1 & -1 & & -1 & 6 & -1 & -1 & -1 \\ & & & & -1 & -1 & -1 & 6 & -1 & -1 \\ & & & & & & -1 & -1 & 3 & -1 \\ & & & -1 & & & -1 & -1 & -1 & 4 \end{bmatrix}$$

The symmetric Laplacian L_s

Weighting schemes

$$\delta_i = \frac{\sum_{j \in N(i)} w_{ij} \left(\mathbf{v}_i - \mathbf{v}_j \right)}{\sum_{j \in N(i)} w_{ij}}$$



- Ignore geometry

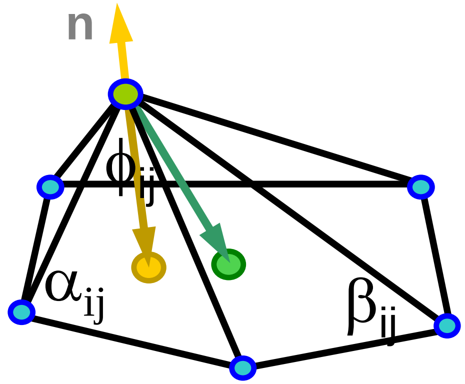
  $\delta_{\textbf{umbrella}} : w_{ij} = 1$

- Integrate over circle around vertex

  $\delta_{\textbf{mean value}} : w_{ij} = \tan \phi_{ij}/2 + \tan \phi_{ij+1}/2$

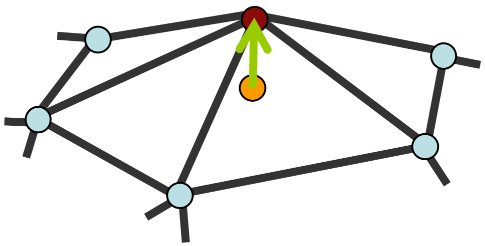
- Integrate over Voronoi region of vertex

  $\delta_{\textbf{cotangent}} : w_{ij} = \cot \alpha_{ij} + \cot \beta_{ij}$

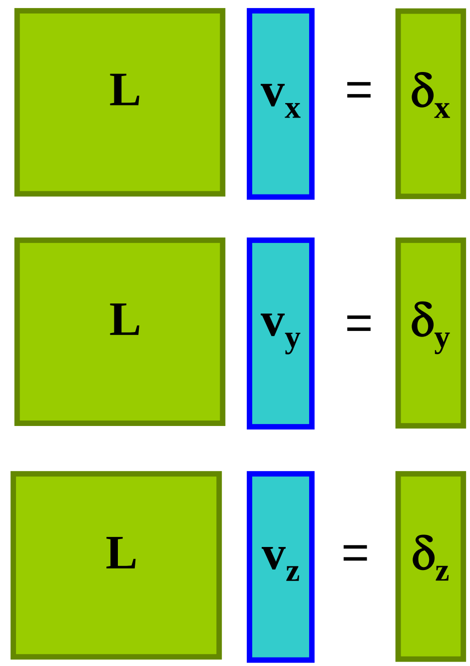


Laplacian mesh

- Vertex positions are represented by Laplacian coordinates (δ_x δ_y δ_z)

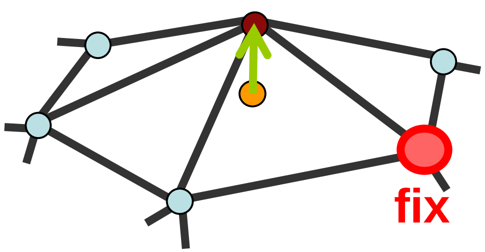


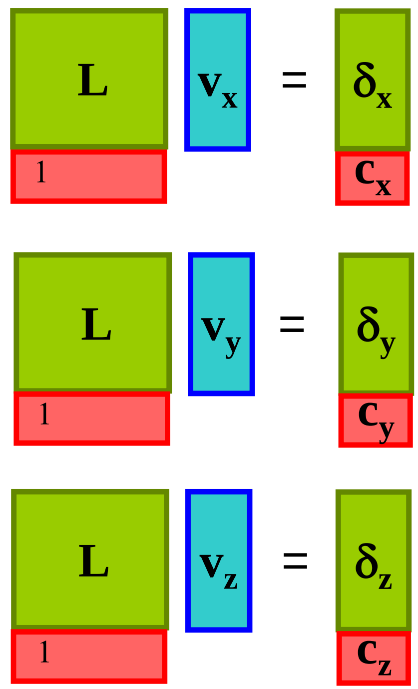
$$\boldsymbol{\delta}_i = \sum_{j \in N(i)} w_{ij} \left(\mathbf{v}_i - \mathbf{v}_j \right)$$



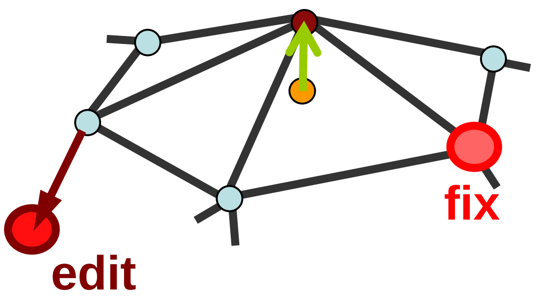
Basic properties

- $\mathrm{rank}(L) = n - c$ ($n - 1$ for connected meshes)
- We can reconstruct the xyz geometry from δ up to translation





Reconstruction

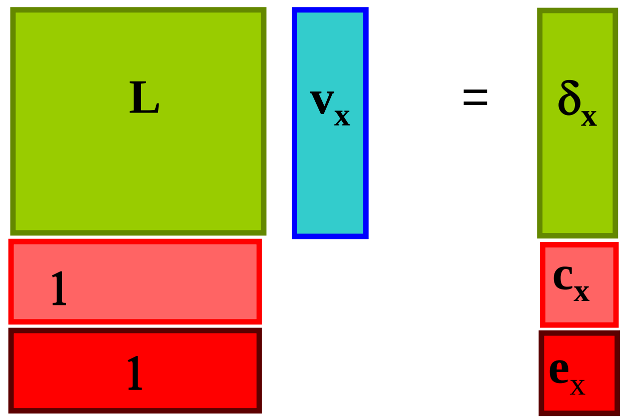


$$\begin{bmatrix} L \\ 1 \\ 1 \end{bmatrix} v_x = \begin{bmatrix} \delta_x \\ c_x \\ e_x \end{bmatrix}$$

$$\begin{bmatrix} L \\ 1 \\ 1 \end{bmatrix} v_y = \begin{bmatrix} \delta_y \\ c_y \\ e_y \end{bmatrix}$$

$$\begin{bmatrix} L \\ 1 \\ 1 \end{bmatrix} v_z = \begin{bmatrix} \delta_z \\ c_z \\ e_z \end{bmatrix}$$

Reconstruction



$$\boxed{\tilde{\mathbf{x}} = \arg\min_{\mathbf{x}} \left(\left\| L\mathbf{x} - \boldsymbol{\delta}_x \right\|^2 + \sum_{s=1}^{k} \left| x_k - c_k \right|^2 \right)}$$

Reconstruction

$$\begin{bmatrix} L \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} v_x \end{bmatrix} = \begin{bmatrix} \delta_x \\ c_x \\ e_x \end{bmatrix}$$

$$A \; x = b$$

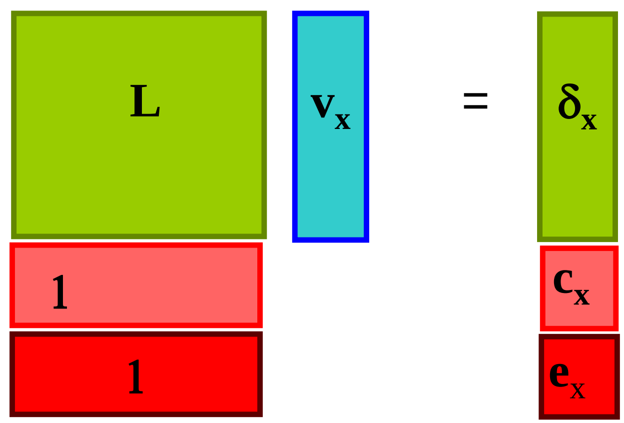
Normal Equations:

$$A^T A \; x = A^T b$$

$$x = \underbrace{(A^T A)^{-1} \; A^T}_{\text{compute once}} b$$

Cool underlying idea

- Mesh vertex positions are defined by minimizer of an objective function



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What we have so far

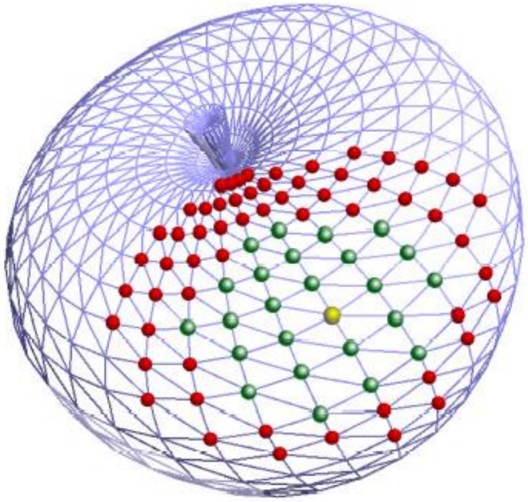
- Laplacian coordinates δ
  - Local representation
  - Translation-invariant
- Linear transition from δ to xyz
  - can constrain more that 1 vertex
  - least-squares solution

Editing using differential coordinates

- The editing process from the user's point of view:

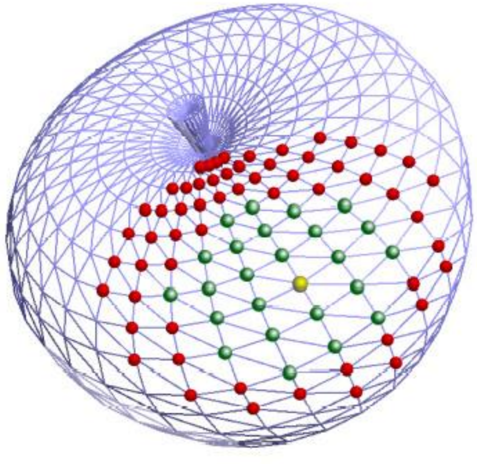
1) First, a ROI , anchors and a handle vertex should be set.

2) Then the edit is Performed By moving this vertex.

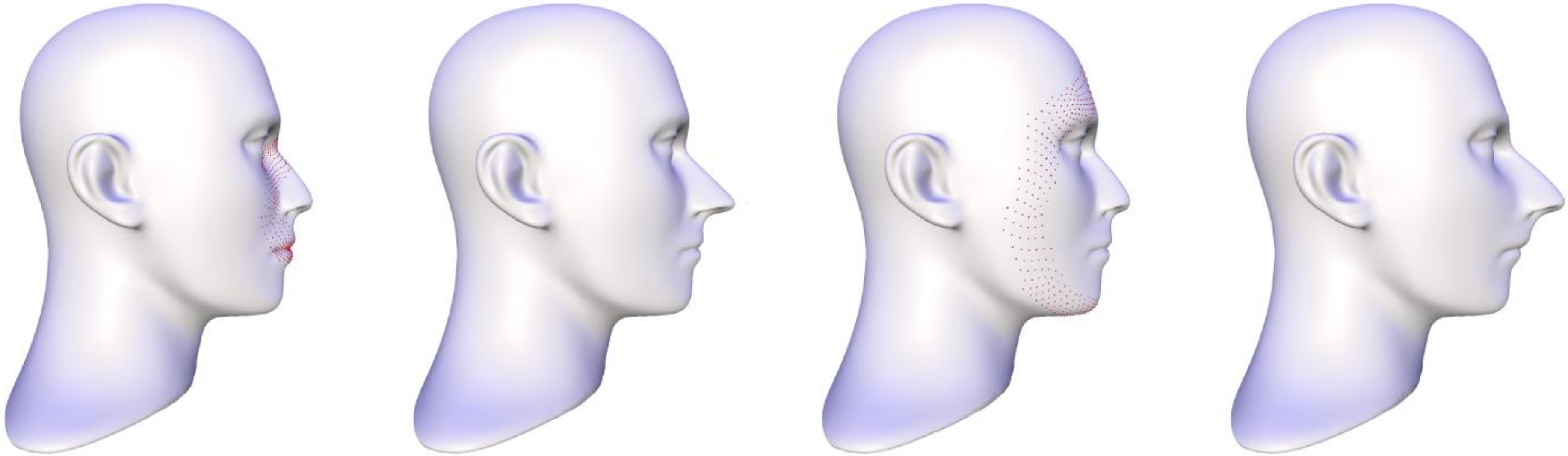


Editing using differential coordinates

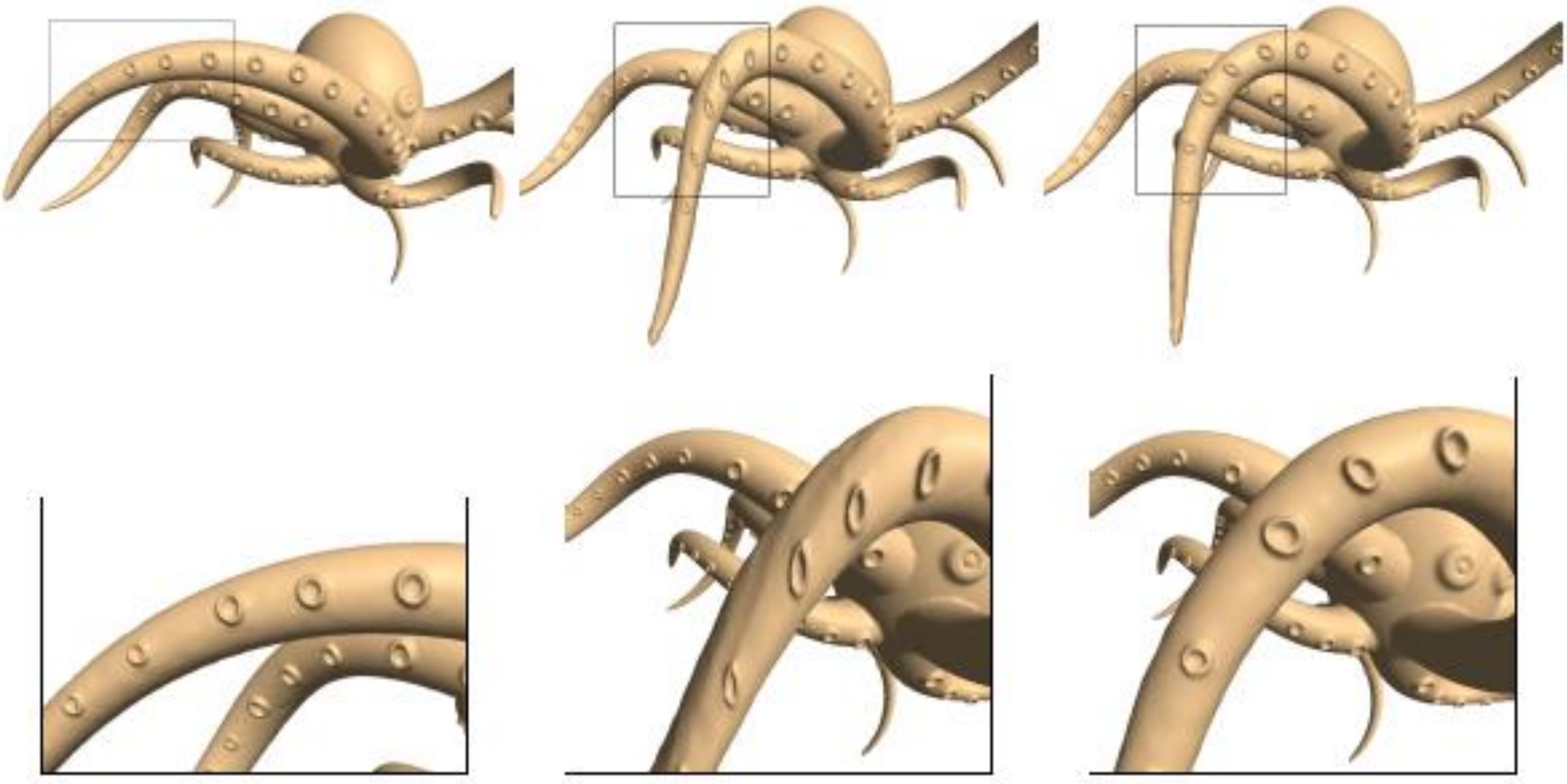
- The user moves the handle and interactively the surface changes.
- The stationary anchors are responsible for smooth transition of the edited part to the rest of the mesh.
- This is done using increasing weight with geodesic distance in the soft spatial equations.



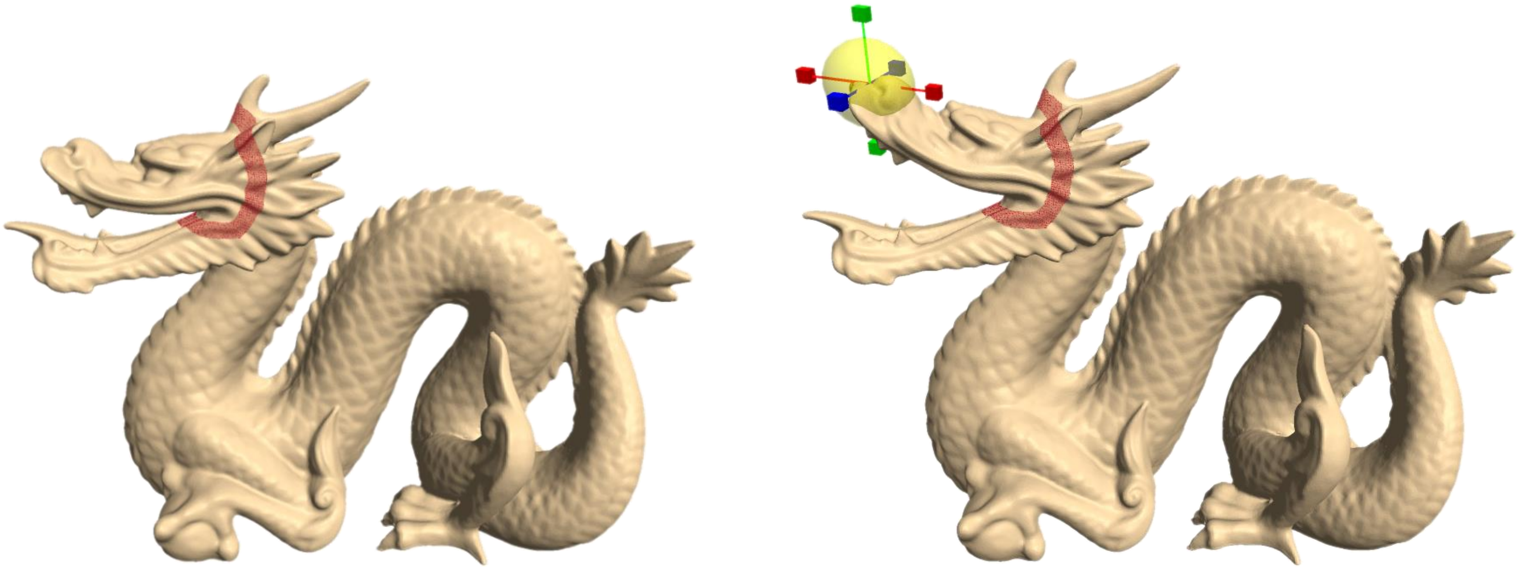
Mesh Editing Example



Mesh Editing Example



Mesh Editing Example



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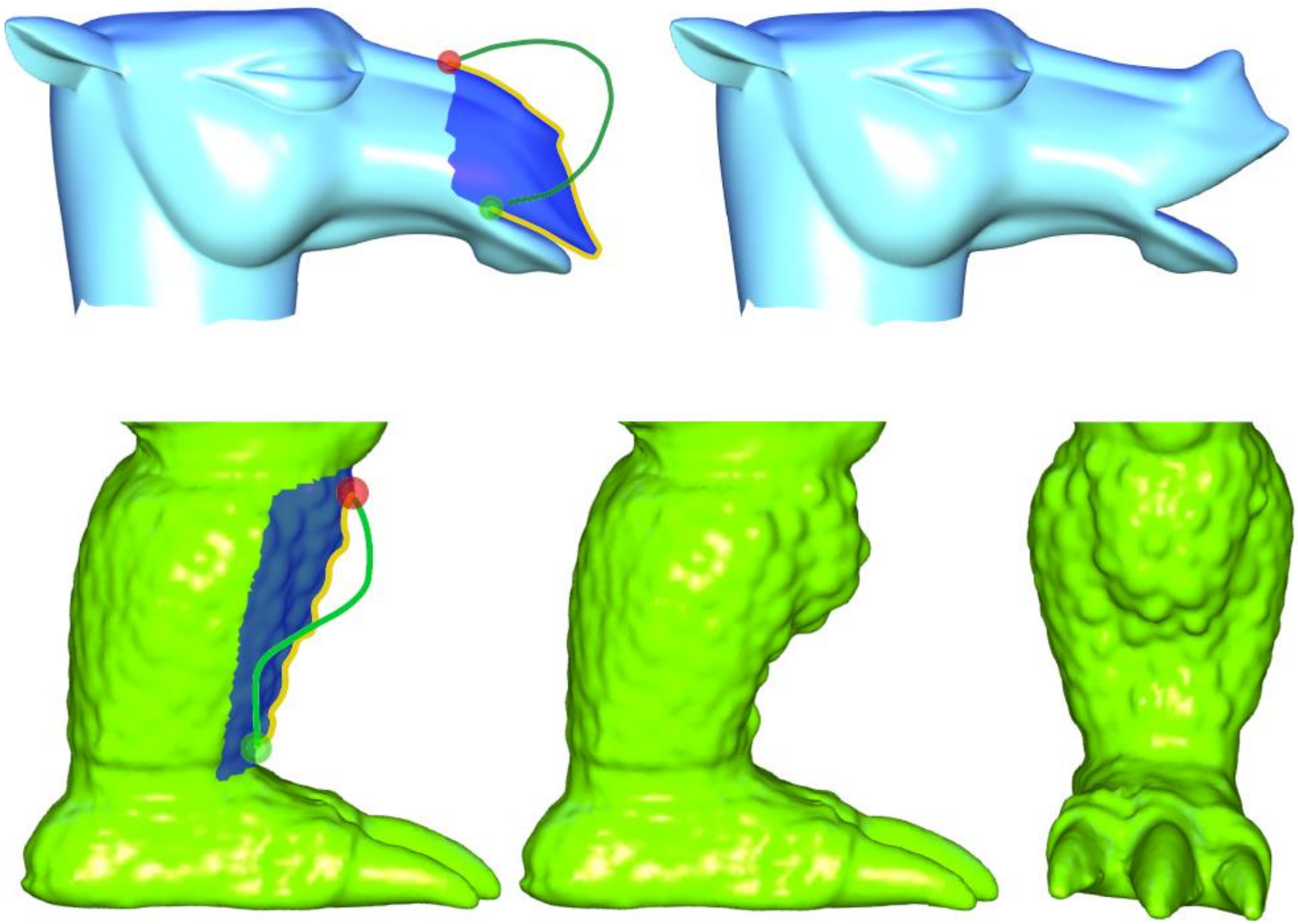
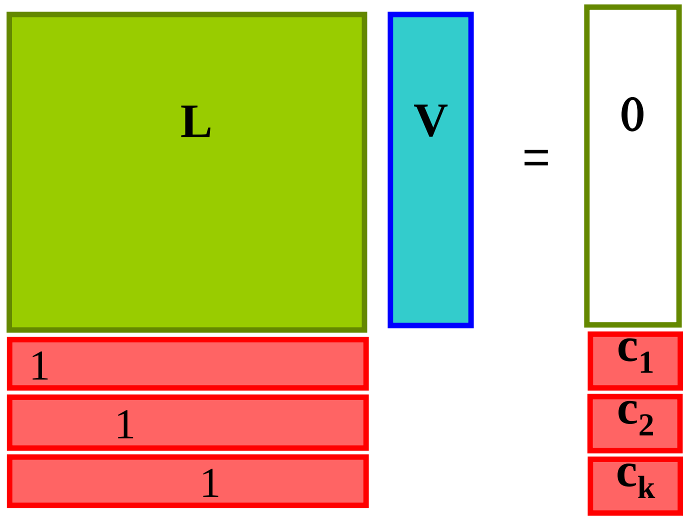


What else can we do with it?

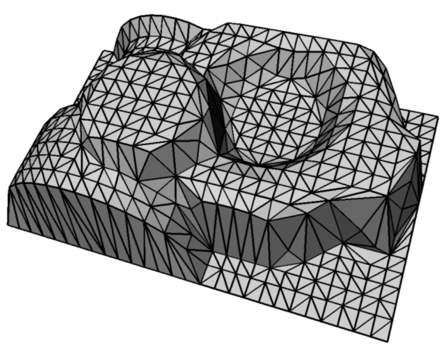
-

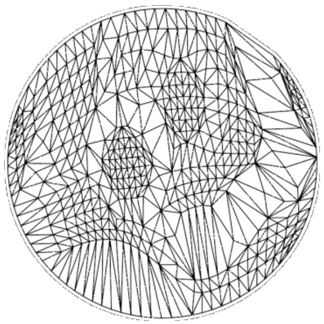
Parameterization

- Use zero Laplacians.

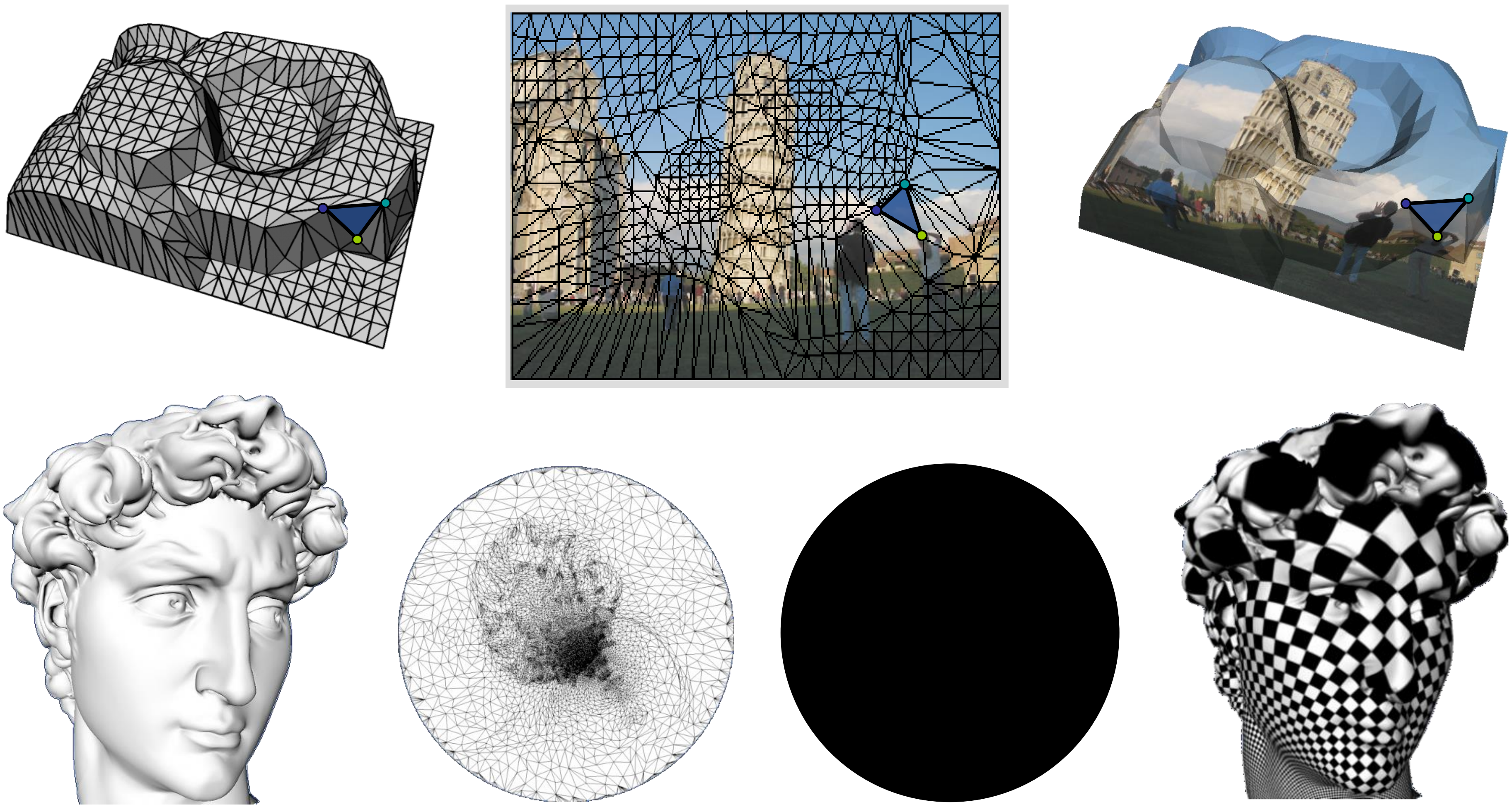


In 2D:





Texture Mapping



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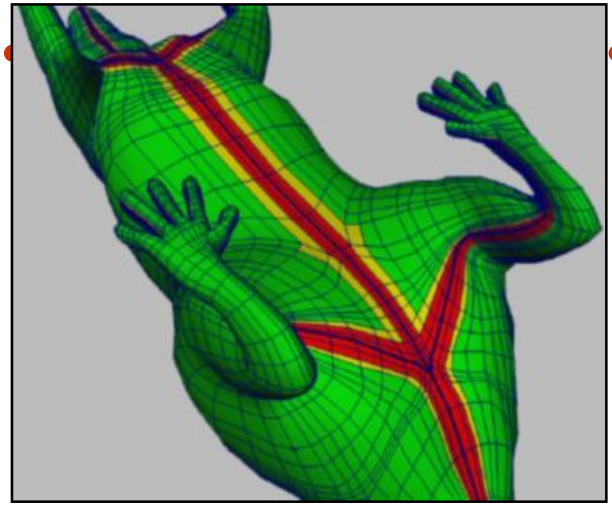

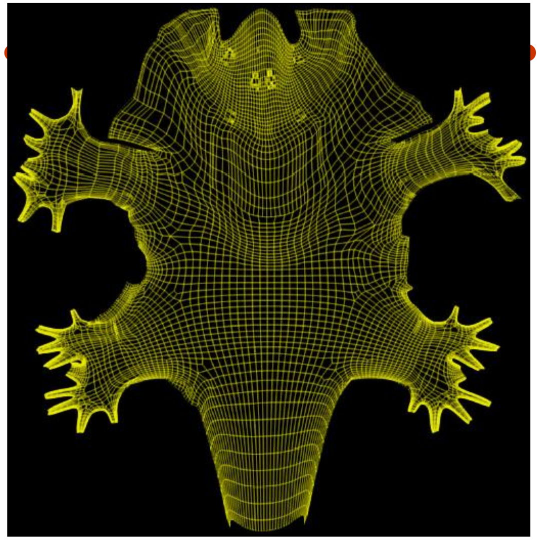

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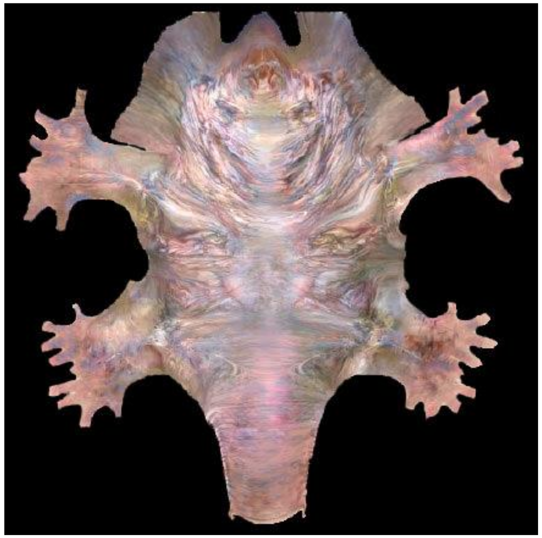
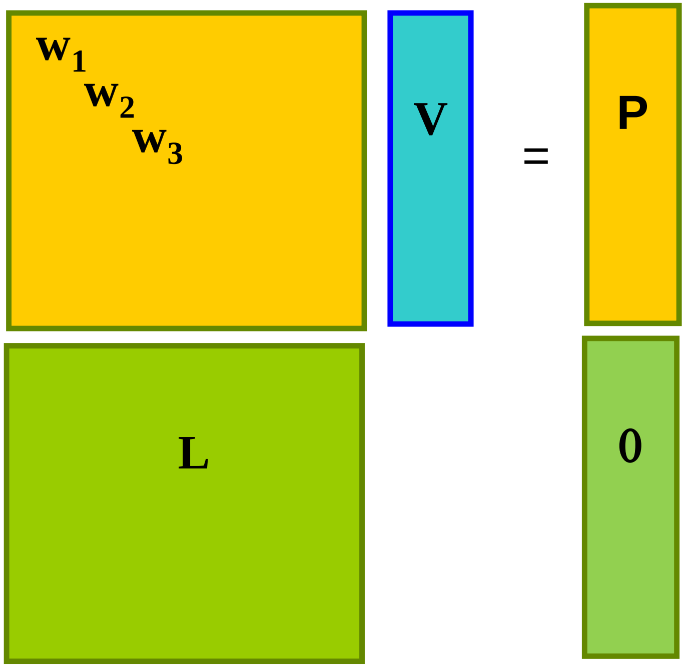


Feature Preserving Smoothing

- Weighted positional and smoothing constraints



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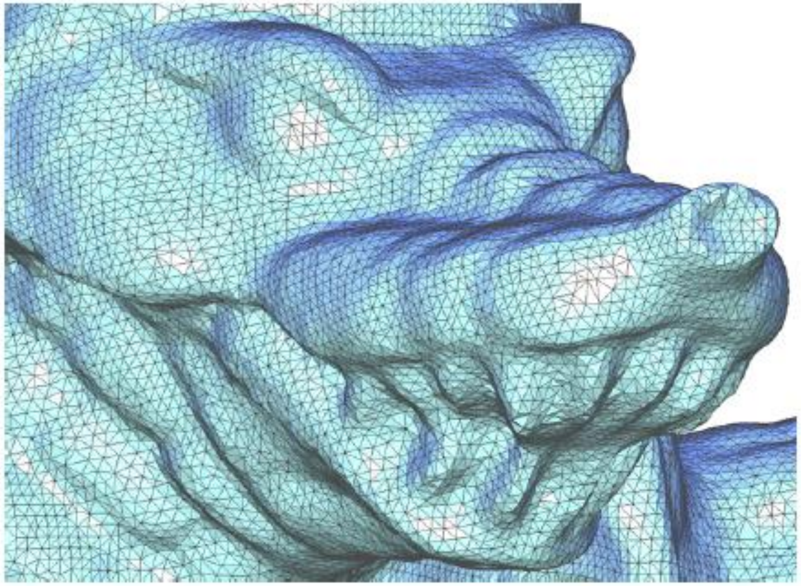
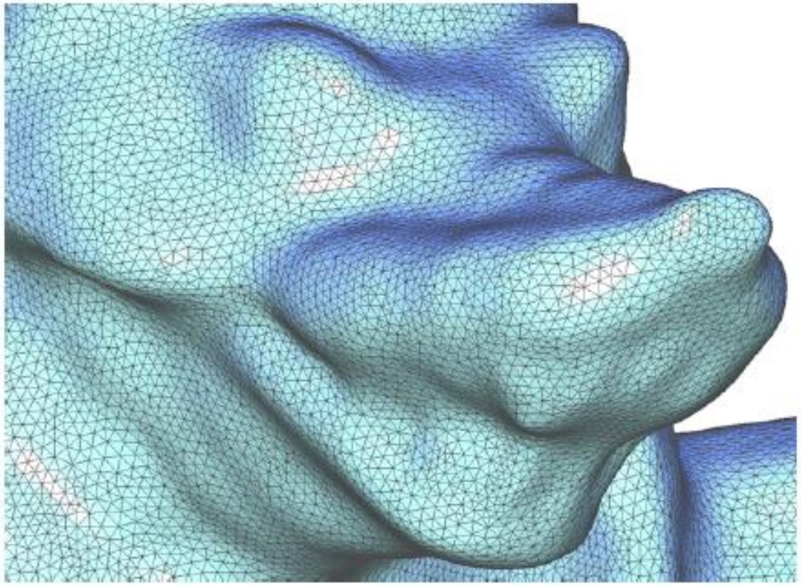


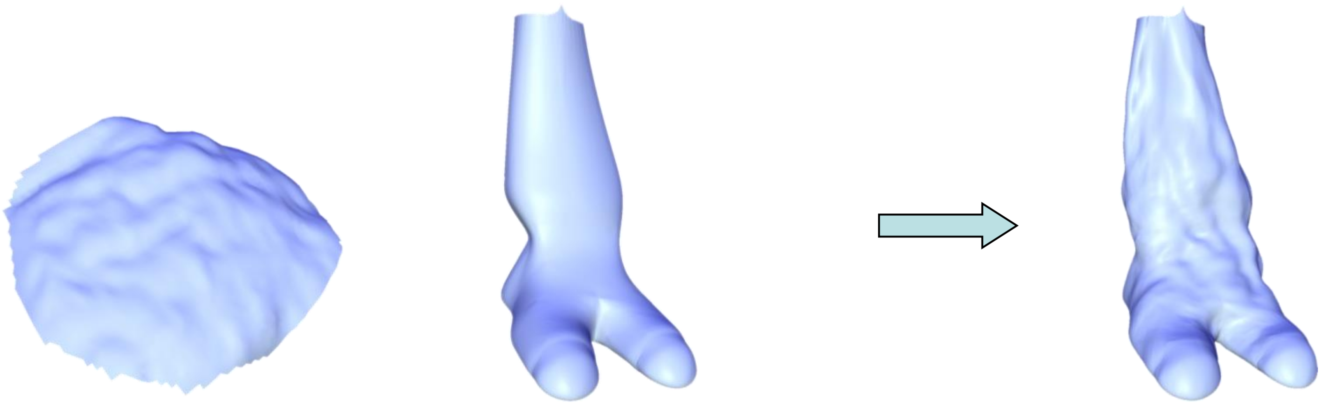
Original



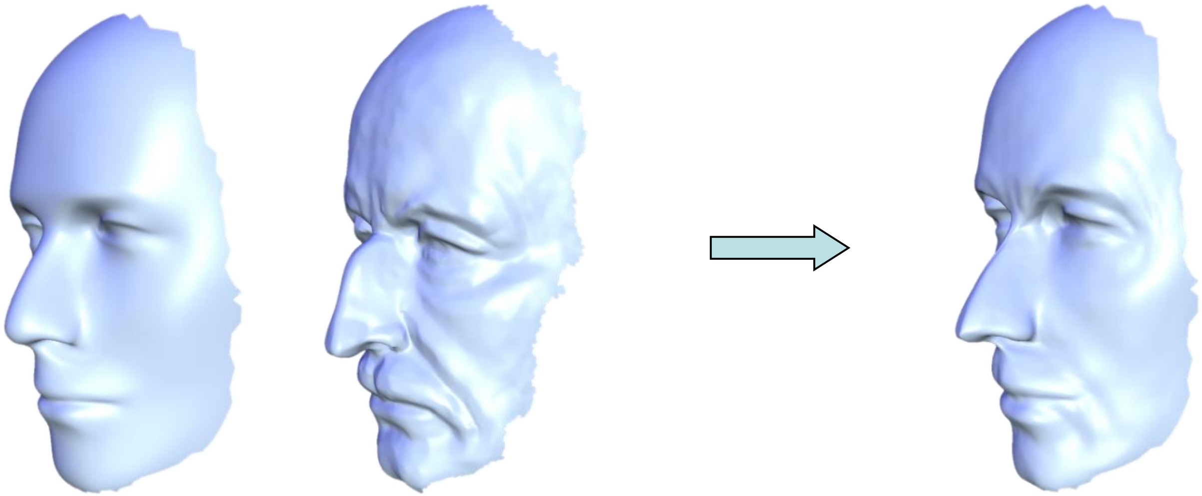
Smoothed

Detail transfer

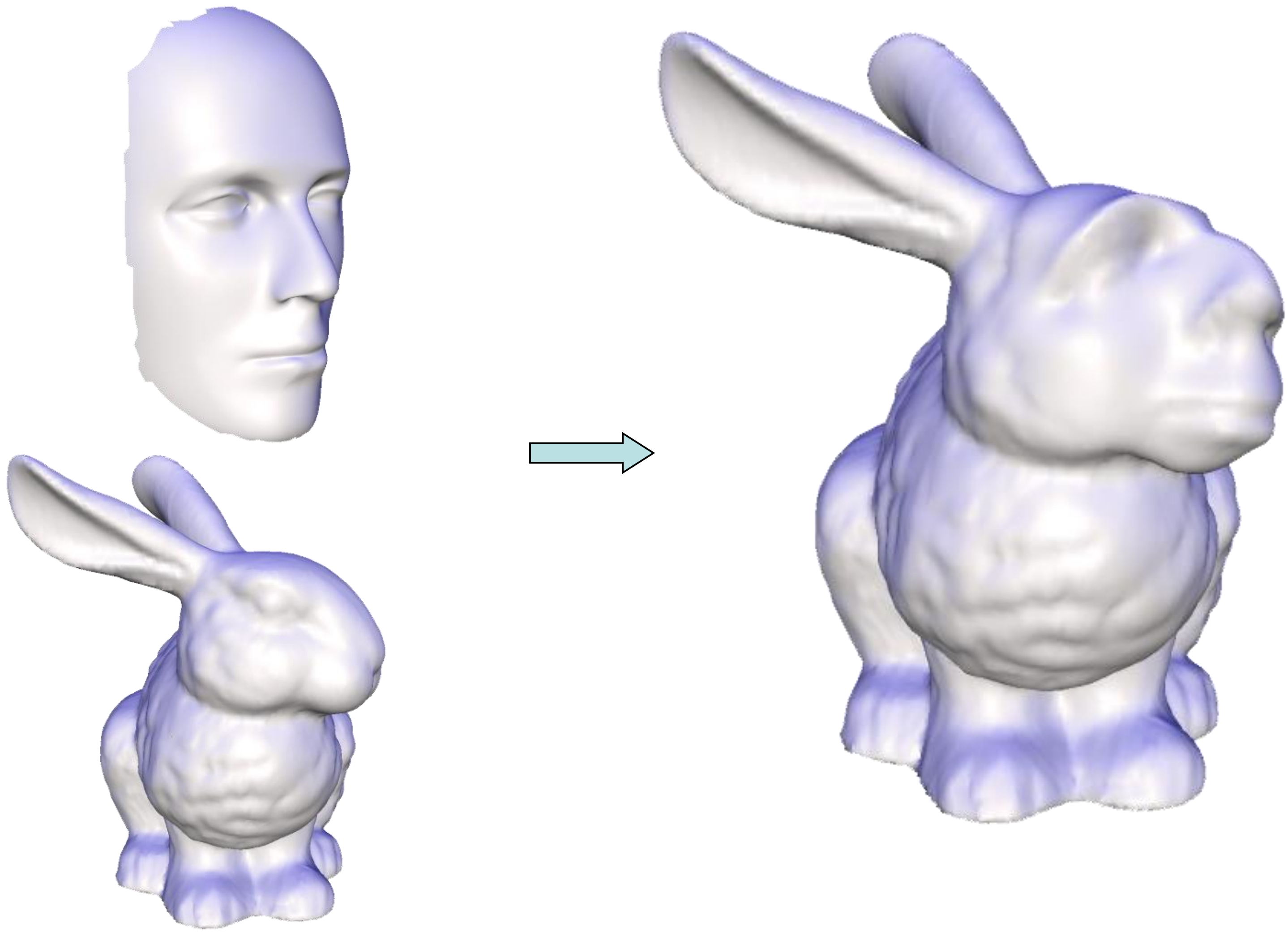
- “Peel“ the coating of one surface and transfer to another



Detail transfer

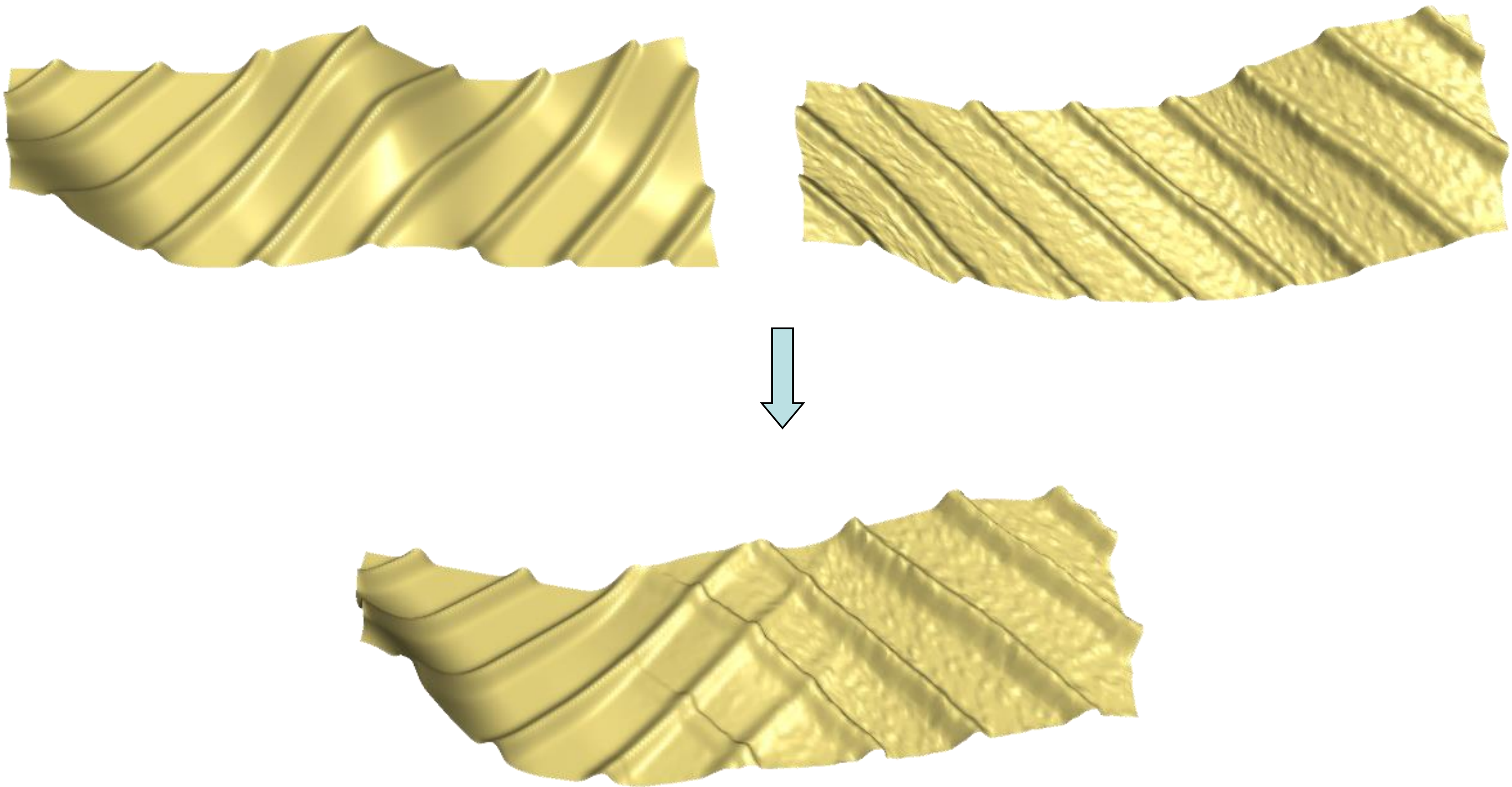


Detail transfer



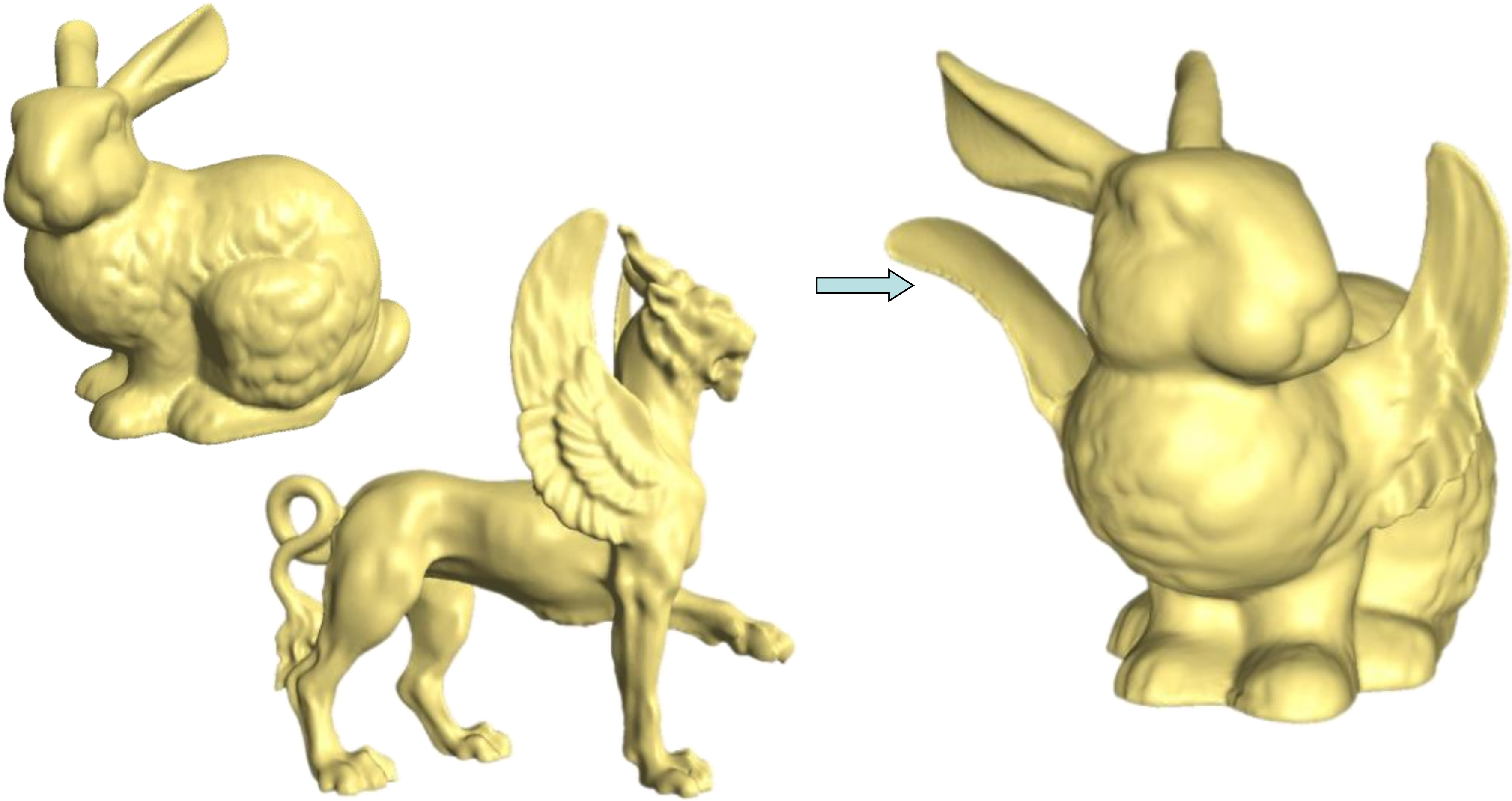
Mixing Laplacians

- Taking weighted average of δ_i and δ'_i



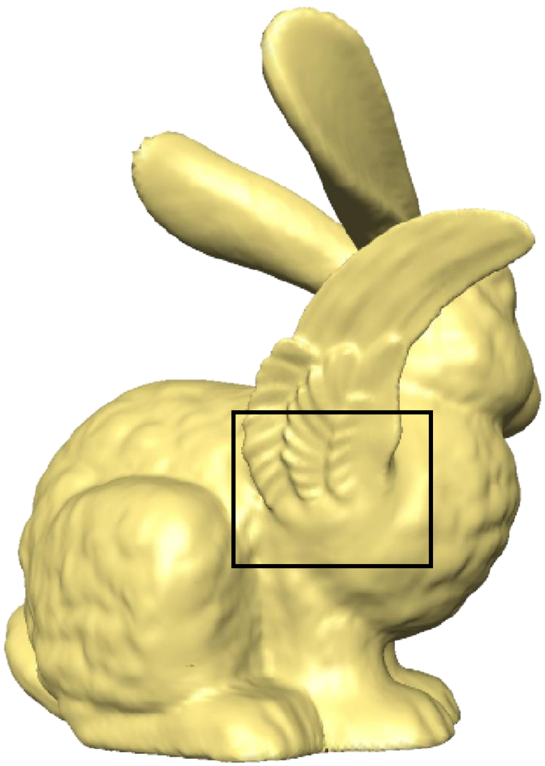
Mesh transplanting

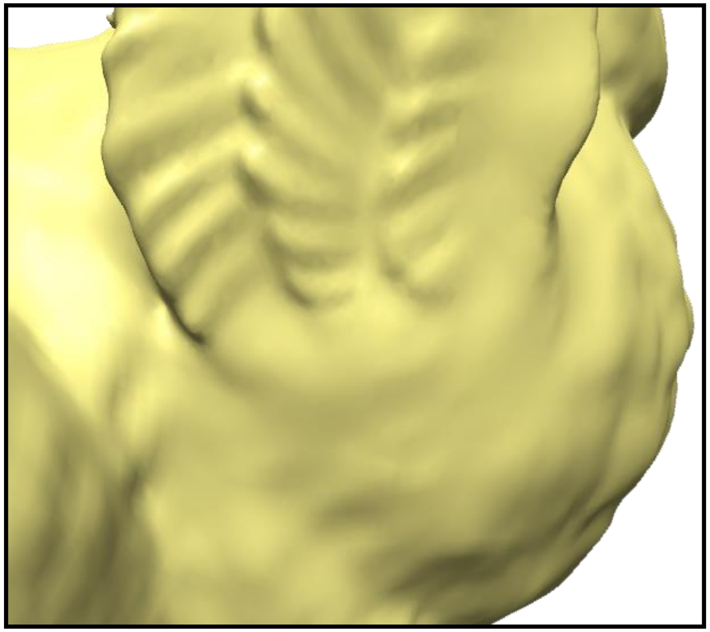
- Geometrical stitching via Laplacian mixing



Mesh transplanting

- Details gradually change in the transition area





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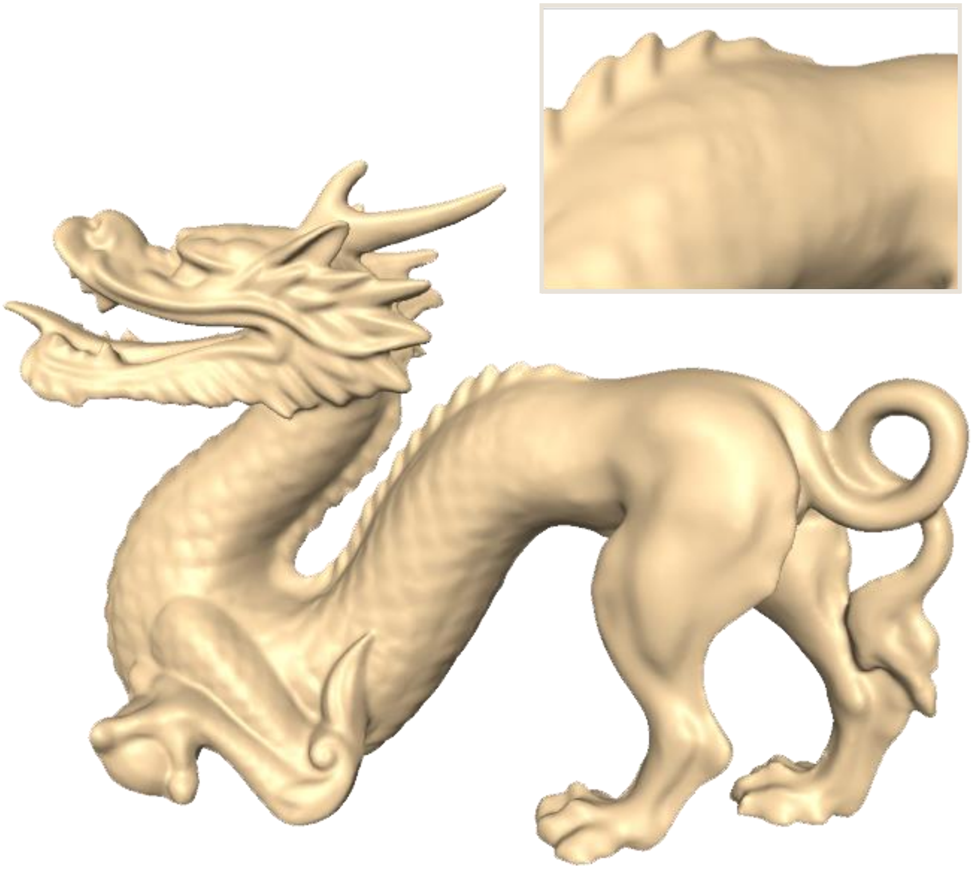

The End