1. Prove von Neumann’s min max theorem. You can assume LP duality.

2. (Braess’s paradox; well-known to transportation planners) Figure (a) depicts a simple network of roads (each is one-way for simplicity) from point $s$ to $t$. The number on the edge is the time to traverse that road. When we say the travel time is $x$, we mean that the time scales linearly with the amount of traffic in it.

![Figure 1: Braess’s paradox](image)

One unit of traffic (a large number of individual drivers) need to travel from $s$ to $t$. (Actually assume is it just a tiny bit less than one unit.) Each driver’s choice of route can be seen as a move in a multiplayer game. What is the Nash equilibrium and what is each driver’s travel time to $t$ in this equilibrium?

Figure (b) depicts the same network with a new superfast highway constructed from $v$ to $w$. What is the new Nash equilibrium and the new travel time?

3. Show that approximating the number of simple cycles within a factor 100 in a directed graph is NP-hard. (Hint: Show that if there is a polynomial-time algorithm for this task, then we can solve the Hamiltonian cycle problem in directed graphs, which is NP-hard. Here the exact constant 100 is not important, and can even be replaced by, say, $n$.)

4. (Extra credit) (Sudan’s list decoding) Let $(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n) \in F^2$ where $F = GF(q)$ and $q \gg n$. We say that a polynomial $p(x)$ describes $k$ of these pairs if $p(a_i) = b_i$ for $k$ values of $i$. This question concerns an algorithm that recovers $p$ even if $k < n/2$ (in other words, a majority of the values are wrong).
(a) Show that there exists a bivariate polynomial $Q(z, x)$ of degree at most $\lceil \sqrt{n} \rceil + 1$ in $z$ and $x$ such that $Q(b_i, a_i) = 0$ for each $i = 1, \ldots, n$. Show also that there is an efficient (poly($n$) time) algorithm to construct such a $Q$.

(b) Show that if $R(z, x)$ is a bivariate polynomial and $g(x)$ a univariate polynomial then $z - g(x)$ divides $R(z, x)$ iff $R(g(x), x)$ is the 0 polynomial.

(c) Suppose $p(x)$ is a degree $d$ polynomial that describes $k$ of the points. Show that if $d$ is an integer and $k > (d + 1)(\lceil \sqrt{n} \rceil + 1)$ then $z - p(x)$ divides the bivariate polynomial $Q(z, x)$ described in part (a). (Aside: Note that this places an upper bound on the number of such polynomials. Can you improve this upper bound by other methods?)

(There is a randomized polynomial time algorithm due to Berlekamp that factors a bivariate polynomial. Using this we can efficiently recover all the polynomials $p$ of the type described in (c). This completes the description of Sudan’s algorithm for list decoding.)

5. (Extra Credit) Show that the simple SDP for Balanced Separator (the one without triangle inequality constraint) is a very poor approximation to the optimum on the cycle graph. (Just $n$ nodes connected in a simple undirected cycle.)