Homework 4

Out: *Nov 19* Due: *Dec 4*

1. Consider a set of n objects (images, songs etc.) and suppose somebody has designed a *distance* function $d(\cdot)$ among them where $d(i,j)$ is the distance between objects i and j. We are trying to find a geometric realization of these distances. Of course, exact realization may be impossible and we are willing to tolerate a factor 2 approximation. We want n vectors $u_1, u_2, \ldots, u_n$ such that $d(i,j) \leq |u_i - u_j|_2 \leq 2d(i,j)$ for all pairs i, j. Describe a polynomial-time algorithm that determines whether such u_i's exist.

2. The course webpage links to a grayscale photo. Interpret it as an $n \times m$ matrix and run SVD on it. What is the value of k such that a rank k approximation gives a reasonable approximation (visually) to the image? What value of k gives an approximation that looks high quality to your eyes? Attach both pictures and your code. (In matlab you need mat2gray function.) Extra credit: Try to explain from first principles why SVD works for image compression at all.

3. Suppose we have a set of n images and for some multiset E of image pairs we have been told whether they are *similar* (denoted +edges in E) or *dissimilar* (denoted $-$edges). These ratings were generated by different users and may not be mutually consistent (in fact the same pair may be rated as $+$ as well as $-$). We wish to *partition* them into clusters $S_1, S_2, S_3, \ldots$ so as to maximise:

   (# of +edges that lie within clusters) + (# of $-$edges that lie between clusters).

   Show that the following SDP is an upperbound on this, where $w^+(ij)$ and $w^-(ij)$ are the number of times pair i, j has been rated $+$ and $-$ respectively.

$$\max \sum_{(i,j)\in E} w^+(ij)(x_i \cdot x_j) + w^-(ij)(1 - x_i \cdot x_j)$$
$$|x_i|_2^2 = 1 \qquad \forall i$$
$$x_i \cdot x_j \geq 0 \qquad \forall i \neq j.$$

4. For the problem in the previous question describe a clustering into 4 clusters that achieves an objective value 0.75 times the SDP value. (Hint: Use Goemans-Williamson style rounding but with two random hyperplanes instead of one. You may need a quick matlab calculation just like GW.)

5. Suppose you are given m halfspaces in $\Re^n$ with rational coefficients. Describe a polynomial-time algorithm to find the largest *sphere* that is contained inside the polyhedron defined by these halfspaces.

6. Let f be an n-variate convex function such that for every x, every eigenvalue of $\nabla^2 f(x)$ lies in $[m, M]$. Show that the optimum value of f is lowerbounded by $f(x) - \frac{1}{2m}|\nabla f(x)|_2^2$ and upperbounded by $f(x) - \frac{1}{2M}|\nabla f(x)|_2^2$, where x is any point. In other words, if the gradient at x is small, then the value of f at x is near-optimal. (Hint: By the mean value theorem, $f(y) = f(x) + \nabla f(x)^T(y-x) + \frac{1}{2}(y-x)^T \nabla^2 f(z)(y-x)$, where z is some point on the line segment joining x, y.)