Homework 3

Out: *Oct 23* Due: *Nov 10*

1. Compute the mixing time (both upper and lower bounds) of a graph on $2n$ nodes that consists of two complete graphs on n nodes joined by a single edge. (Hint: Use elementary probability calculations and reasoning about "probability fluid"; no need for eigenvalues.)

2. Let M be the Markov chain of a 5-regular undirected graph that is connected. Each node has self-loops with probability 1/2. We saw in class that 1 is an eigenvalue with eigenvector $\vec{1}$. Show that every other eigenvalue has magnitude at most $1 - 1/10n^2$. (Hint: First show that a connected graph cannot have 2 eigenvalues that are 1.) What does this imply about the mixing time for a random walk on this graph from an arbitrary starting point?

3. This question will study how mixing can be much slower on directed graphs. Describe an n-node directed graph (with max indegree and outdegree at most 5) that is fully connected but where the random walk takes $\exp(\Omega(n))$ time to mix (and the walk ultimately does mix). Argue carefully.

4. (Game-playing equilibria) Recall the game of Rock, Paper, Scissors. Let's make it quantitative it by saying that the winning player wins \$ 1 whereas the loser gets \$ 0. (In other words, the game is not zero sum.) A draw results in both getting 0. Suppose we make two copies of the multiplicative weight update algorithm to play each other over many iterations. Both start using the uniformly random strategy (i.e., play each of Rock/paper/scissors with probability 1/3) and learn from experience using the MW rule. One imagines that repeated play causes them to converge to some kind of *equilibrium.* (a) Predict by just calculation/introspection what this equilibrium is. (Be honest; it's Ok to be wrong!). (b) Run this experiment on Matlab or any other programming environment and report what you discovered and briefly explain it. (We'll discuss the result in class.)

5. Describe an example (i.e., an appropriate set of n points in $\Re^n$) that shows that the Johnson-Lindenstrauss dimension reduction method — the transformation described in Lecture, with an appropriate scaling— does *not* preserve ℓ_1 distances within even factor 2. (Extra credit: Show that no *linear transformation* suffices, let alone J-L.)

6. (Dimension reduction for SVM's with margin) Suppose we are given two sets P, N of unit vectors in $\Re^n$ with the guarantee that there exists a hyperplane $a \cdot x = 0$ such that every point in P is on one side and every point in N is on the other. Furthermore, the ℓ_2 distance of each point in P and N to this hyperplane is at least ϵ. Then show using the Johnson Lindenstrauss lemma (hint: you can use it as a black box) that a random linear mapping to $O(\log n/\epsilon^2)$ dimensions and such that the points are still separable by a hyperplane with margin $\epsilon/2$.

7. Implement the portfolio management appearing in the notes for Lecture 11 in any programming environment and check its performance on S& P stock data (download from http://ocobook.cs.princeton.edu/links.htm). Include your code as well as the final performance (i.e., the percentage gain achieved by your strategy).

8. (Extra credit) Calculate the eigenvectors and eigenvalues of the n-dimensional boolean hypercube, which is the graph with vertex set $\{-1, 1\}^n$ and x, y are connected by an edge iff they differ in exactly one of the n locations. (Hint: Use symmetry extensively.)