COS 226
Midterm Review
Fall 2015

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Time and location:

- The midterm is during lecture on
  - **Tuesday, Oct 27th from 11-12:20pm.**
- The exam will start and end promptly, so please do arrive on time.
- The midterm room is
  - McDonnell A01 - P03 P03A - all 11am precepts
  - Jadwin 10 - P01 P05 - 9am and 1:30pm precepts
  - Friend 101 - P02 P02A P04 P04A - all the even precepts (10am and 12:30pm)
- Failure to go to the right room can result in a serious deduction on the exam. There will be no makeup exams except under extraordinary circumstances, which must be accompanied by the recommendation of a Dean.

Rules

- Closed book, closed note.
- You may bring one 8.5-by-11 sheet (one side) with notes in your own handwriting to the exam.
- No electronic devices (including calculators, laptops, and cell phones).

Materials covered

- *Algorithms, 4th edition, Sections 1.3–1.5, Chapter 2, and Chapter 3.*
- *Lectures 1–11*
- *Programming assignments 1–5.*

concepts (so far) in a nutshell

List of algorithms and data structures:

quick-find	quick-union	weighted quick-union	
resizing arrays	linked lists	stacks	queues
insertion sort	selection sort	Knuth shuffle	
mergesort	bottom-up mergesort		
quicksort	3-way quicksort	quickselect	
binary heaps	heapsort		
sequential search	binary search	BSTs	
kd-trees	interval search trees		
2-3 trees	left-leaning red-black BSTs		

Sorting Invariants

An Invariant

- An invariant is a statement that is TRUE throughout the execution of an algorithm
  - Insertion sort
    - A[0..0] is sorted at the beginning (pre-condition)
    - A[0..i] sub-array is sorted at any intermediate point (loop-invariant)
    - A[0..N-1] sorted at the end (post-condition)

Question?

1. Which sorting algorithms have the invariant A[0…i] sub-array is sorted after i-steps?
   1. Insertion sort
   2. Selection sort
   3. Top-down mergesort
   4. Bottom-up mergesort
   5. 2-way quicksort
   6. 3-way quicksort
   7. Heapsort
   8. Knuth Shuffle

Question?

2. Which sorting algorithms have the invariant A[i+1…N-1] sub-array is the original after i-steps?

1. Insertion sort
2. Selection sort
3. Top-down mergesort
4. Bottom-up mergesort
5. 2-way quicksort
6. 3-way quicksort
7. Heapsort
8. Knuth Shuffle

Question?

3. Which sorting algorithms have the invariant A[0…i] are the smallest in the array after i-steps?

1. Insertion sort
2. Selection sort
3. Top-down mergesort
4. Bottom-up mergesort
5. 2-way quicksort
6. 3-way quicksort
7. Heapsort
8. Knuth Shuffle

Question?

4. Which sorting algorithms have the invariant A[0] is the largest in the array after i-steps?

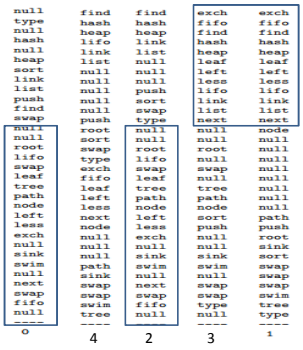
1. Insertion sort
2. Selection sort
3. Top-down mergesort
4. Bottom-up mergesort
5. 2-way quicksort
6. 3-way quicksort
7. Heapsort
8. Knuth Shuffle

Question?

5. Which sorting algorithms have the invariant A[0..i] is a rearrangement and A[i+1..N-1] is the original in the array after i-steps?

1. Insertion sort
2. Selection sort
3. Top-down mergesort
4. Bottom-up mergesort
5. 2-way quicksort
6. 3-way quicksort
7. Heapsort
8. Knuth Shuffle

Insertion, selection or mergesort?



Further invariants:
2. Insertion: A[i+1…N-1] is the original
3. Selection: A[0…i] are the smallest in the array and are in the final position
4. Mergesort: A[0..i] is sorted and I = 2^k for some k

2-way and 3-way quicksort

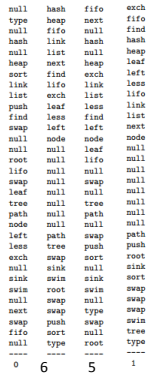
The pivot (first entry) is placed in the final position

2-way - All elements to the left are <= and right are >=

3-way - All elements equal to pivot are in the middle. Left is < and right is >

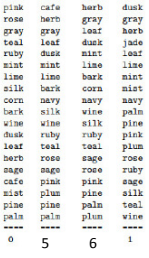
- How can one recognize 2-way from 3-way if first entry has no duplicates?
  - **2-way** does not swap a lot and do one swap of the pivot at the end. That is, all elements <= pivot must stay in their original position and all elements >= must stay in their original position
  - **3-way** takes the pivot with it and making swaps. All elements < have moved to the left as pivot moves forward

2-way or 3-way?

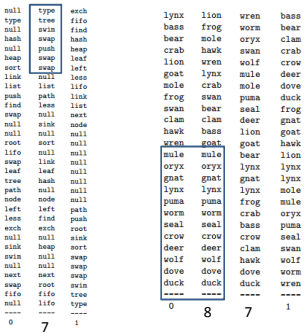


5. 2-way quicksort

6. 3-way quicksort



Heapsort and Knuth Shuffle



Invariants
7. Heapsort :
A[1..N] is a max-heap.
In fact A[1..i] is a max-heap for any i

8. Knuth shuffle
A[0..i] is randomized and A[i+1..N-1] is the original

Analysis of Algorithms

Question

True or False

I. Tilde notation includes the coefficient of the highest order term.

II. Tilde notation provides both an upper bound and a lower bound on the growth of a function.

III. Big-Oh notation suppresses lower order terms, so it does not necessarily accurately describe the behavior of a function for small values of N.

Count operations

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      for (int k = j+1; k < N; k++)
         if (a[i] + a[j] >= a[k]) count++;
```

Suppose that it takes 1 second to execute this code fragment when $N = 1000$. Using tilde notation, formulate a hypothesis for the running time (in seconds) of the code fragment as a function of N.

Analysis of Algorithms

- Performance of an algorithm is measured using
  - comparisons, array accesses, exchanges,
  - memory requirements
- best, worst, average
  - Performance measure based on some specific input type
    - Eg: sorted, random, reverse sorted. Almost sorted

Examples

$f(N) \sim g(N)$ means $\lim_{N \to \infty} \frac{f(N)}{g(N)} = 1$

notation	provides	example	shorthand for	used to
Big Theta	asymptotic order of growth	$\Theta(N^2)$	$\frac{1}{2} N^2$ $10 N^2$ $5 N^2 + 22 N \log N + 3N$ ⋮	classify algorithms
Big Oh	$\Theta(N^2)$ and smaller	$O(N^2)$	$10 N^2$ $100 N$ $22 N \log N + 3 N$ ⋮	develop upper bounds
Big Omega	$\Theta(N^2)$ and larger	$\Omega(N^2)$	$\frac{1}{2} N^2$ N^5 $N^3 + 22 N \log N + 3 N$ ⋮	develop lower bounds

Amortized analysis

- Measure of average performance over a series of operations
  - Some good, few bad
  - Overall good (or not bad)
- Array resizing
  - Resize by 1 as extra space is needed
  - Resize by doubling the size as extra space is needed
  - Resize by tripling the size as extra space is needed

Amortized analysis

- Resize by 1 as needed
  - Start with N=1, 1W (new)
  - N=2, 1W (to copy), 1W(new)
  - N=3, 2W (to copy), 1W (new)
  - Total writes (to write N elements)
    - 1+2+… + (N-1) + (1+ 1 + 1 + ..+1) ~ N^2
    - Cost per operation ~ N (expensive)
- Resize by doubling
  - Start with N=1, 1W (new)
  - N=2, 1W (to copy), 1W(new)
  - N=4, 2W (to copy), 2W (new)
  - N=8, 4W (to copy), 4W (new)
  - Total writes (to write N = 2^k elements for some k)
    - 1+2+… + 2^{k-1} + (1+2+… + 2^{k-1}) ~ N
    - Cost per operation ~ 1 (cheap)

useful formulas

$1 + 2 + \ldots + N.$ $\quad \sum_{i=1}^{N} i \sim \int_{x=1}^{N} x\,dx \sim \frac{1}{2}N^2$

$1^k + 2^k + \ldots + N^k.$ $\quad \sum_{i=1}^{N} i^k \sim \int_{x=1}^{N} x^k dx \sim \frac{1}{k+1}N^{k+1}$

$1 + 1/2 + 1/3 + \ldots + 1/N.$ $\quad \sum_{i=1}^{N} \frac{1}{i} \sim \int_{x=1}^{N} \frac{1}{x}dx = \ln N$

3-sum triple loop. $\quad \sum_{i=1}^{N}\sum_{j=i}^{N}\sum_{k=j}^{N} 1 \sim \int_{x=1}^{N}\int_{y=x}^{N}\int_{z=y}^{N} dz\,dy\,dx \sim \frac{1}{6}N^3$

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

$\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i = 2$

$\int_{x=0}^{\infty}\left(\frac{1}{2}\right)^x dx = \frac{1}{\ln 2} \approx 1.4427$

counting memory

- standard data types (int, bool, double)
- object overhead – 16 bytes
- array overhead – 24 bytes
- references – 8 bytes
- Inner class reference – 8 bytes (unless inner class is static)

```
public class TwoThreeTree<Key extends Comparable<Key>, Value> {
   private Node root;

   private class Node {
      private int count;                  // subtree count
      private Key key1, key2;             // the one or two keys
      private Value value1, value2;       // the one or two values
      private Node left, middle, right;   // the two or three subtrees
   }
   ...
}
```

- How much memory is needed for a 2-3 tree object that holds N nodes? Express the answer in tilde notation, big O notation

Algorithm and Data Structure Design

Design problems

- Typically challenging
- There can be many possible solutions
  - partial credit awarded
- Usually it is a data structure design to meet certain performance requirements for a given set of operations
  - Example, create a data structure that meets the following performance requirements
    - findMedian in ~1 , insert ~ lg n, delete ~ lg n
  - Example: A leaky queue that can remove from any point, that can insert to end and delete from front, all in logarithmic time or better
- Typical cues to look for
  - *log n* time may indicate that you need a sorted array or balanced BST or some sort of a heap
  - Amortized time may indicate, you can have some costly operations once in a while, but on average, it must perform as expected

design problem #1

- Design a randomizedArray structure that can insert and delete a random Item from the array. Need to guarantee amortized constant performance
  - Insert(Item item)
  - delete()

design problem #2

An ExtrinsicMaxPQ is a priority queue that allows the programmer to specify the priority of an object independent of the intrinsic properties of that object. This is unlike the MaxPQ from class, which assumed the objects were comparable and used the compare method to establish priority. You may assume the Items are comparable.

```
public class ExtrinsicMaxPQ<Item extends Comparable<Item>> {
      ExtinsicMaxPQ()                  //do not implement
      void put(Item x, int priority)
      Item delMax()
}
```

If an Item already exists in the priority queue, then its priority is changed instead of adding another item. All operations should complete in **amortized logarithmic time in the worst case**. Your ExtrinsicMaxPQ should use **memory proportional to the number of items**. **For a small amount of partial credit, you may assume that no Item's priority is ever changed (i.e. no item is inserted twice).**

Example:

```
put("cat", 12)      // cat is inserted with priority 12
put("dog", 10)
put("swimp is a raccoon who enjoys fries and does not like to eat dirt", 11)
put("dog", 15)      // dog's priority is changed to 15
delMax()            // deletes dog, which has priority 15, cat is now max
put("fish", 20)     // fish is inserted, and is now max
put("fish", 11)     // fish's priority is reduced to 11, cat is again max
delMax()            // removes cat (priority 12), either swimp or fish now max
delMax()            // removing either swimp or fish is OK, both priority 11
```

Design Problem #3

`public class MoveToFront<Item>`

`MoveToFront()`	*create an empty move-to-front data structure*
`void add(Item item)`	*add the item at the front (index 0) of the sequence (thereby increasing the index of every other item)*
`Item itemAtIndex(int i)`	*the item at index i*
`void mtf(int i)`	*move the item at index i to index 0 (thereby increasing the index of items 0 through i – 1)*

All operations should take time proportional to $\log N$ *in the worst case, where* N *is the number of items in the data structure.*

Key choices for design problems

- Choice of data structure
  - LLRB
    - insert, delete, rank, update, find, max, min, rank (logarithmic time)
  - Hash table
    - insert, find, update (constant time)
  - Heap
    - delMax, insert (logarithmic)
  - symbol table
    - ordered (same as LLRB), unordered (same as hash table)
  - LIFO Stack and FIFO queue
    - inserts and deletes in amortized constant time

Bonus Slides

Possible/impossible questions

Possible/impossible questions

- We can build a heap in linear time. Is it possible to build a BST in linear time?
- is it possible to find the max or min of any list in log N time?
- Is it possible to create a collection where an item can be stored or found in constant time
- Is it possible to design a max heap where find findMax, insert and delMax can be done in constant time?

Possible/impossible questions

- is it possible to sort a list of n keys in linear time, where only d (some small constant) distinct keys exists among n keys?
- Is it possible to find median in an unsorted list in linear time?

Possible/impossible questions

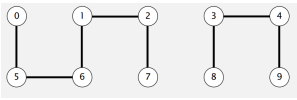
- is it possible to implement a FIFO queue using a single array, still have amortized constant time for enque and deque?
- Is it possible to solve the 3-sum problem in n log n time?

Why?

- Why do we ever use a BST when we can always use a hash table?
- Why do we ever use arrays when we can use linked lists?
- Why do we ever use a heap when we can always use a LLRB?
- Why do we ever use a LLRB when we can always randomize and get a balance BST?

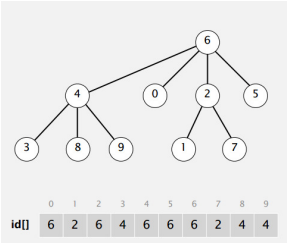
Union-find

quick-union and quick-find



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

Weighted quick-union



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

logarithmic union and find performance

- **Maintain heuristics**
  - when merging two trees, smaller one gets attached to larger one – height does not increase
  - Height only increase when two trees are the same size

Weighted Union-find question

Circle the letters corresponding to `id[]` arrays that *cannot* possibly occur during the execution of the weighted quick union algorithm.

		0	1	2	3	4	5	6	7	8	9
A.	a[i]:	8	0	4	0	0	4	0	4	2	0
B.	a[i]:	4	1	8	2	1	5	1	1	4	5
C.	a[i]:	3	3	6	9	3	6	3	4	1	9
D.	a[i]:	2	1	1	1	1	1	1	2	1	7

- What is the right approach to solving this?
  - Be sure that it is a tree (no cycles)
  - Be sure that the size of the tree is lg N

Answer to union-find question

Circle the letters corresponding to `id[]` arrays that *cannot* possibly occur during the execution of the weighted quick union algorithm.

```
              0  1  2  3  4  5  6  7  8  9
            ------------------------------
A.   a[i]:  8  0  4  0  0  4  0  4  2  0

B.   a[i]:  4  1  8  2  1  5  1  1  4  5

C.   a[i]:  3  3  6  9  3  6  3  4  1  9

D.   a[i]:  2  1  1  1  1  1  1  2  1  7
```

A B C

A. The `id[]` array contains a cycle: $8 \rightarrow 2 \rightarrow 4 \rightarrow 0 \rightarrow 8$.

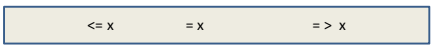
B. The height of the forest is $4 > \lg(10)$.

C. The size of tree rooted at the parent of 3 is less than twice the size of tree rooted at 3.

D. The following sequence of union operations would create the given `id[]` array:
`2-0 1-8 7-9 0-9 8-5 4-1 1-9 3-8 5-6`

Extra Slides

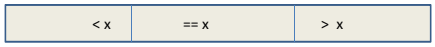
Demo of 2-way quick sort



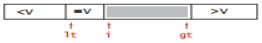
I M I W R F D T T O S D E E P

3-way quick sort

- same as 2-way quicksort
- works well with duplicate keys
- same process
  - choose a pivot, say x
  - partition the array as follows



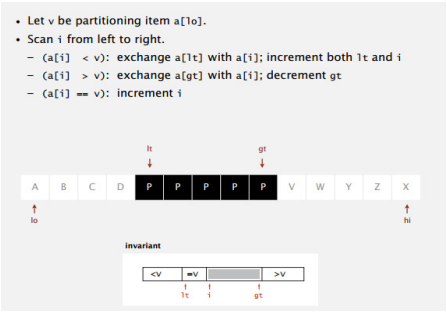
  - Invariant



- uses Dijkstra's 3-way partitioning algorithm

3-way partitioning demo

- Let v be partitioning item a[lo].
- Scan i from left to right.
  - (a[i] < v): exchange a[lt] with a[i]; increment both lt and i
  - (a[i] > v): exchange a[gt] with a[i]; decrement gt
  - (a[i] == v): increment i

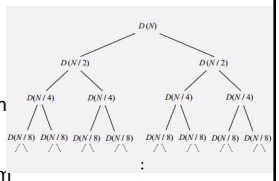


Demo of 3-way quick sort

I M I W R F D T T O S D E E P

Top-down merge sort

- facts
  - recursive
  - merging is the main operation
- performance
  - merging 2-sorted arrays takes lin time
  - merge sort tree is of height lg N
  - consistent linearithmic algorithm
- other properties
  - uses extra linear space
  - Stable
    - equal keys retain relative position in subsequent sorts



Properties
- Left most items get sorted first
- Look for 2-sorted, 4-sorted etc

bottom-up merge sort

- facts
  - iterative
  - merges sub-arrays of size 2, 4, 8 (lg N times) to finally get a sorted array
- performance
  - merging all sub arrays takes linear time in each step
  - merge continues lg N times
  - consistent linearithmic algorithm
- other properties
  - no extra space
  - stable
    - merge step retains the position of the equal keys

Properties
- Look for 2-sorted, 4-sorted, 8-sorted etc

Heap Sort

- build a max heap
- delete max and insert into the end of the array (if heap is implemented as an array) until heap is empty
- performance is linearithmic

Knuth shuffle

- Generates random permutations of a finite set
- algorithm

```
for (int i=n-1; i > 0; i--) {
    j = random(0..i);
    exch(a[j], a[i]);
}
```

Priority Queues

Binary heaps

- Invariant
  - for each node N
    - Key in N >= key in left child and key in right child (order invariant)
    - A complete binary tree – all levels are full except perhaps the last level. Elements are added to the last level from L to R
- good logarithmic performance for
  - Insert(Item item)
  - delMax()
  - max()
- heap building
  - bottom-up ➔ linear time (sink each level)
  - top-down ➔ linearithmic (insert and swim)

Heap questions

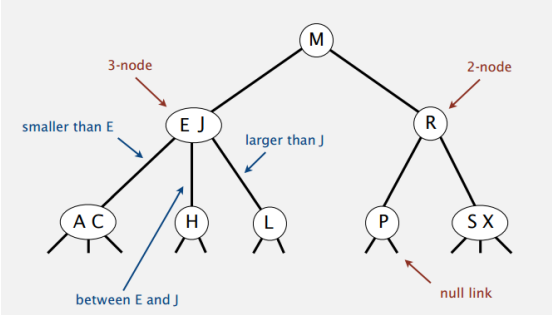
- Given a heap, find out which key was inserted last?
  - it must be along the path of the right most leaf node in the tree
  - We always delete the root by exchanging that with the last leaf node
- Build a heap
  - Bottom-up
  - Top-down
- Applications
  - can be used in design questions where delete, insert takes logarithmic time and find max takes constant time

Ordered Symbol Tables

	sequential search	binary search	BST
search	N	$\log N$	h
insert	N	N	h
min / max	N	1	h
floor / ceiling	N	$\log N$	h
rank	N	$\log N$	h
select	N	1	h
ordered iteration	$N \log N$	N	N

Balanced Trees

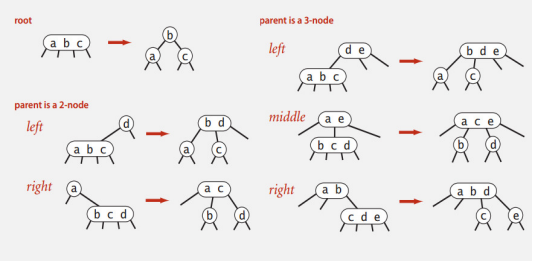
2-3 Trees



Two invariants

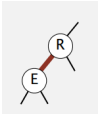
- Balance invariant – each path from root to leaf nodes have the same length
- Order invariant – an inorder traversal of the tree produces an ordered sequence

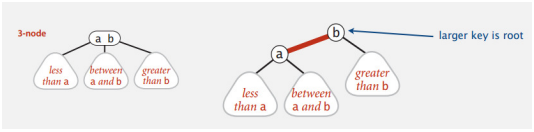
2-3 Tree operations



Red-black trees

- How to represent 3-nodes?
  - Regular BST with red "glue" links.

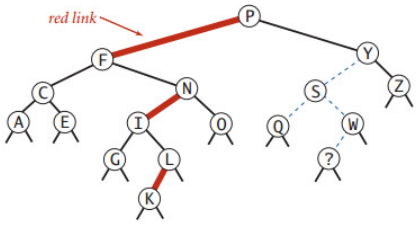




Red-black tree properties

- A BST such that
  - No node has two red links connected to it
  - Every path from root to null link has the same number of **black links**
  - Red links lean left.

examples



Red-black tree questions

- add or delete a key to/from a red-black tree and show how the tree is rebalanced
- Determining the value of an unknown node
  - Less than M, greater than G, less than L
- Know all the operations
  - Left rotation, right rotation, color flip
  - Know how to build a LLRB using operations
- Know how to go from 2-3 tree to a red-black tree and vice versa