6.5 Reductions

- introduction
- designing algorithms
- establishing lower bounds
- classifying problems
- intractability
Overview

Shifting gears.

- From individual problems to problem-solving models.
- From linear/quadratic to polynomial/exponential scale.
- From implementation details to conceptual frameworks.

Goals.

- Place algorithms and techniques we've studied in a larger context.
- Introduce you to important and essential ideas.
- Inspire you to learn more about algorithms!
6.5 REductions

- introduction
- designing algorithms
- establishing lower bounds
- classifying problems
- intractability
### Desiderata. Classify problems according to computational requirements.

<table>
<thead>
<tr>
<th>complexity</th>
<th>order of growth</th>
<th>examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear</td>
<td>( N )</td>
<td>( \text{min, max, median, Burrows-Wheeler transform, ...} )</td>
</tr>
<tr>
<td>linearithmic</td>
<td>( N \log N )</td>
<td>( \text{sorting, element distinctness, closest pair, Euclidean MST, ...} )</td>
</tr>
<tr>
<td>quadratic</td>
<td>( N^2 )</td>
<td>( ? )</td>
</tr>
<tr>
<td></td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>exponential</td>
<td>( c^N )</td>
<td>( ? )</td>
</tr>
</tbody>
</table>
Desiderata. Classify problems according to computational requirements.

Desiderata'. Suppose we could (could not) solve problem $X$ efficiently. What else could (could not) we solve efficiently?

“Give me a lever long enough and a fulcrum on which to place it, and I shall move the world.” — Archimedes
Def. Problem $X$ reduces to problem $Y$ if you can use an algorithm that solves $Y$ to help solve $X$.

Cost of solving $X = \text{total cost of solving } Y + \text{cost of reduction}$.

- Perhaps many calls to $Y$ on problems of different sizes (typically only 1 call).
- Preprocessing and postprocessing (typically less than cost of solving $Y$).
**Reduction**

**Def.** Problem $X$ **reduces to** problem $Y$ if you can use an algorithm that solves $Y$ to help solve $X$.

**Ex 1:** element distinctness reduces to sorting
To solve element distinctness on $N$ items:
- Sort $N$ items.
- Check adjacent pairs for equality.

**Cost of element distinctness.** $N \log N + N$. 
**Reduction**

**Def.** Problem $X$ reduces to problem $Y$ if you can use an algorithm that solves $Y$ to help solve $X$.

---

**Ex 2:** finding the median reduces to sorting.

To find the median of $N$ items:

- Sort $N$ items.
- Return item in the middle.

Cost of finding the median. $N \log N + 1$. [even though we know how to do it better]
**Reduction**

**Def.** Problem \( X \) reduces to problem \( Y \) if you can use an algorithm that solves \( Y \) to help solve \( X \).

**Beware of novice error.** Confusing \( X \) reduces to \( Y \) with \( Y \) reduces to \( X \).
Reductions: quiz 1

Which of the following reductions have we encountered in this course?

I. MAX-FLOW reduces to MIN-CUT.
II. MIN-CUT reduces to MAX-FLOW.

A. I only.
B. II only.
C. Both I and II.
D. Neither I nor II.
E. I don't know.
6.5 REDUCTIONS

- introduction
- designing algorithms
- establishing lower bounds
- classifying problems
- intractability
Reduction: design algorithms

**Def.** Problem $X$ reduces to problem $Y$ if you can use an algorithm that solves $Y$ to help solve $X$.

**Design algorithm.** Given an algorithm for $Y$, can also solve $X$.

**More familiar reductions.**
- Mincut reduces to maxflow.
- Arbitrage reduces to negative cycles.
- Bipartite matching reduces to maxflow.
- Seam carving reduces to shortest paths in a DAG.
- Burrows-Wheeler transform reduces to suffix sort.
  ...

**Mentality.** Since I know how to solve $Y$, can I use that algorithm to solve $X$?

*programmer's version: I have code for $Y$. Can I use it for $X$?*
3-collinear

3-COLLINEAR. Given \( N \) distinct points in the plane, are there 3 (or more) that all lie on the same line?

Brute force \( N^3 \). For all triples of points \((p, q, r)\), check if they are collinear.
3-collinear reduces to sorting

**Sorting-based algorithm.** For each point $p$,
- Compute the slope that each other point $q$ makes with $p$.
- Sort the $N - 1$ points by slope.
- Collinear points are adjacent.

Cost of solving **3-COLLINEAR.** $N^2 \log N + N^2$. 

![Diagram showing the computation of slopes and sorting of points.](attachment:image.png)
Proposition. Undirected shortest paths (with nonnegative weights) reduces to directed shortest path.

Proof. Replace each undirected edge by two directed edges.

Cost of solving undirected shortest paths. \( E \log V + (E + V) \).
Some reductions in combinatorial optimization

baseball elimination

mincut

bipartite matching

undirected shortest paths (nonnegative)

directed shortest paths (nonnegative)

assignment problem

maxflow

directed shortest paths (no neg cycles)

linear programming

linear programming

linear programming

shortest paths (in a DAG)

seam carving

arbitrage

directed shortest paths (nonnegative)

undirected shortest paths (nonnegative)
Some reductions in string processing

circular shift

longest repeated substring

substring search

longest common substring

longest palindromic substring

tandem repeats

suffix trees

suffix arrays

Burrows–Wheeler transform

Lempel–Ziv decomposition
6.5 Reductions

- introduction
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- intractability
**Goal.** Prove that a problem requires a certain number of steps.

**Ex.** In decision tree model, any compare-based sorting algorithm requires $\Omega(N \log N)$ compares in the worst case.

**Bad news.** Very difficult to establish lower bounds from scratch.

**Good news.** Spread $\Omega(N \log N)$ lower bound to $Y$ by reducing sorting to $Y$.  

*assuming cost of reduction is not too high*
Linear-time reductions

**Def.** Problem \( X \) linear-time reduces to problem \( Y \) if \( X \) can be solved with:

- Linear number of standard computational steps.
- Constant number of calls to \( Y \).

**Establish lower bound:**

- If \( X \) takes \( \Omega(N \log N) \) steps, then so does \( Y \).
- If \( X \) takes \( \Omega(N^2) \) steps, then so does \( Y \).

**Mentality.**

- If I could easily solve \( Y \), then I could easily solve \( X \).
- I can’t easily solve \( X \).
- Therefore, I can't easily solve \( Y \).
Reductions: quiz 2

Which of the following reductions is not a linear-time reduction?

A. **ELEMENT-DISTINCTNESS** reduces to **SORTING**.

B. **MIN-CUT** reduces to **MAX-FLOW**.

C. **3-COLLINEAR** reduces to **SORTING**.

D. **BURROWS-WHEELER-TRANSFORM** reduces to **SUFFIX-SORTING**.

E. *I don't know.*
**ELEMENT-DISTINCTNESS** linear-time reduces to **2D-CLOSEST-PAIR**

**ELEMENT-DISTINCTNESS.** Given $N$ elements, are any two equal?

**2D-CLOSEST-PAIR.** Given $N$ points in the plane, find the closest pair.

| 590584  |
| -23439854 |
| 1251432  |
| -2861534  |
| 3988818  |
| -43434213 |
| 333255   |
| 13546464  |
| 89885444  |
| -43434213 |
| 11998833  |

**element distinctness**

**2d closest pair**
**ELEMENT-DISTINCTNESS** linear-time reduces to 2D-CLOSEST-PAIR

**ELEMENT-DISTINCTNESS.** Given $N$ elements, are any two equal?

**2D-CLOSEST-PAIR.** Given $N$ points in the plane, find the closest pair.

**Proposition.** **ELEMENT-DISTINCTNESS** linear-time reduces to 2D-CLOSEST-PAIR.

**Pf.**

- **ELEMENT-DISTINCTNESS instance:** $x_1, x_2, \ldots, x_N$.
- **2D-CLOSEST-PAIR instance:** $(x_1, x_1), (x_2, x_2), \ldots, (x_N, x_N)$.
- The $N$ elements are distinct iff distance of closest pair $> 0$.

**ELEMENT-DISTINCTNESS lower bound.** In quadratic decision tree model, any algorithm that solves **ELEMENT-DISTINCTNESS** takes $\Omega(N \log N)$ steps.

**Implication.** In quadratic decision tree model, any algorithm for **2D-CLOSEST-PAIR** takes $\Omega(N \log N)$ steps.
Some linear-time reductions in computational geometry

- element distinctness (N log N lower bound)
- sorting
- 2d closest pair
- 2d convex hull
- 2d Euclidean MST
- Delaunay triangulation
- Voronoi diagram
- largest empty circle (N log N lower bound)
- smallest enclosing circle
Lower bound for 3-COLLINEAR

3-SUM.  Given $N$ distinct integers, are there three that sum to 0?

3-COLLINEAR.  Given $N$ distinct points in the plane, are there 3 (or more) that lie on the same line?
Lower bound for 3-COLLINEAR

**3-SUM.** Given $N$ distinct integers, are there three that sum to 0?

**3-COLLINEAR.** Given $N$ distinct points in the plane, are there 3 (or more) that lie on the same line?

**Proposition.** 3-SUM linear-time reduces to 3-COLLINEAR.

**Pf.** [next two slides]

**Conjecture.** Any algorithm for 3-SUM requires $\Omega(N^{2-\varepsilon})$ steps.

**Implication.** No sub-quadratic algorithm for 3-COLLINEAR likely.
**3-SUM** linear-time reduces to **3-COLLINEAR**

**Proposition.** 3-SUM linear-time reduces to 3-COLLINEAR.

- 3-SUM instance: \(x_1, x_2, \ldots, x_N\).
- 3-COLLINEAR instance: \((x_1, x_1^3), (x_2, x_2^3), \ldots, (x_N, x_N^3)\).

**Lemma.** If \(a, b,\) and \(c\) are distinct, then \(a + b + c = 0\) if and only if \((a, a^3), (b, b^3),\) and \((c, c^3)\) are collinear.
**3-SUM linear-time reduces to 3-COLLINEAR**

**Proposition.** 3-SUM linear-time reduces to 3-COLLINEAR.

- **3-SUM instance:** \( x_1, x_2, \ldots, x_N \).
- **3-COLLINEAR instance:** \((x_1, x_1^3), (x_2, x_2^3), \ldots, (x_N, x_N^3)\).

**Lemma.** If \( a, b, \) and \( c \) are distinct, then \( a + b + c = 0 \)
if and only if \((a, a^3), (b, b^3), \) and \((c, c^3)\) are collinear.

**Pf.** Three distinct points \((a, a^3), (b, b^3), \) and \((c, c^3)\) are collinear iff:

\[
0 = \begin{vmatrix}
  a & a^3 & 1 \\
  b & b^3 & 1 \\
  c & c^3 & 1 \\
\end{vmatrix}
\]

\[
= a(b^3 - c^3) - b(a^3 - c^3) + c(a^3 - b^3)
\]

\[
= (a - b)(b - c)(c - a)(a + b + c)
\]
More geometric reductions and lower bounds

3-SUM
(conjectured $N^{2-\epsilon}$ lower bound)

POLYGONAL-CONTAINMENT
3-COLLINEAR
DIHEDRAL-ROTATION
GEOMETRIC-BASE

3-CONCURRENT
MIN-AREA-TRIANGLE
LINE-SEGMENT-SEPARATOR
PLANAR-MOTION-PLANNING
Complexity of 3-SUM

April 2014. Some recent evidence that the complexity might be $N^{3/2}$.

Threesomes, Degenerates, and Love Triangles*

Allan Grønlund  
MADALGO, Aarhus University  

Seth Pettie  
University of Michigan  

April 4, 2014

Abstract

The 3SUM problem is to decide, given a set of $n$ real numbers, whether any three sum to zero. We prove that the decision tree complexity of 3SUM is $O(n^{3/2} \sqrt{\log n})$, that there is a randomized 3SUM algorithm running in $O(n^2 (\log \log n)^2 / \log n)$ time, and a deterministic algorithm running in $O(n^2 (\log \log n)^{5/3} / (\log n)^{2/3})$ time. These results refute the strongest version of the 3SUM conjecture, namely that its decision tree (and algorithmic) complexity is $\Omega(n^2)$.

---

1 This work is supported in part by the Danish National Research Foundation grant DNF 84 through the Center for Massive Data Algorithmics (MADALGO). S. Pettie is supported by NSF grants CCF-1217338 and CNS-1318294 and a grant from the US-Israel Binational Science Foundation.
Establishing lower bounds: summary

Establishing lower bounds through reduction is an important tool in guiding algorithm design efforts.

Q. How to convince yourself no linear-time \textsc{Euclidean-MST} algorithm exists?
A2. [easy way] Linear-time reduction from element distinctness.

2d Euclidean MST
6.5 Reductions

- introduction
- designing algorithms
- establishing lower bounds
- classifying problems
- intractability
Classifying problems: summary

Desiderata. Problem with algorithm that matches lower bound.
Ex. Sorting and element distinctness have complexity $N \log N$.

Desiderata'. Prove that two problems $X$ and $Y$ have the same complexity.
• First, show that problem $X$ linear-time reduces to $Y$.
• Second, show that $Y$ linear-time reduces to $X$.
• Conclude that $X$ has complexity $N^b$ iff $Y$ has complexity $N^b$ for $b \geq 1$.

even if we don't know what it is

\[ X = \text{sorting} \]
\[ Y = \text{element distinctness} \]
\[ \text{integer multiplication} \]
\[ \text{integer division} \]
Integer arithmetic reductions

**Integer multiplication.** Given two $N$-bit integers, compute their product. **Brute force.** $N^2$ bit operations.

\[
\begin{array}{c}
\phantom{0}1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\times & & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
\hline
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
\]
Integer arithmetic reductions

**Integer multiplication.** Given two $N$-bit integers, compute their product.

**Brute force.** $N^2$ bit operations.

<table>
<thead>
<tr>
<th>problem</th>
<th>arithmetic</th>
<th>order of growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>integer multiplication</td>
<td>$a \times b$</td>
<td>$M(N)$</td>
</tr>
<tr>
<td>integer division</td>
<td>$a / b, \ a \mod b$</td>
<td>$M(N)$</td>
</tr>
<tr>
<td>integer square</td>
<td>$a^2$</td>
<td>$M(N)$</td>
</tr>
<tr>
<td>integer square root</td>
<td>$\lfloor \sqrt{a} \rfloor$</td>
<td>$M(N)$</td>
</tr>
</tbody>
</table>

integer arithmetic problems with the same complexity as integer multiplication

**Q.** Is brute-force algorithm optimal?
History of complexity of integer multiplication

<table>
<thead>
<tr>
<th>year</th>
<th>algorithm</th>
<th>order of growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>?</td>
<td>brute force</td>
<td>$N^2$</td>
</tr>
<tr>
<td>1962</td>
<td>Karatsuba</td>
<td>$N^{1.585}$</td>
</tr>
<tr>
<td>1963</td>
<td>Toom–3, Toom–4</td>
<td>$N^{1.465}$, $N^{1.404}$</td>
</tr>
<tr>
<td>1966</td>
<td>Toom–Cook</td>
<td>$N^{1 + \varepsilon}$</td>
</tr>
<tr>
<td>1971</td>
<td>Schönhage–Strassen</td>
<td>$N \log N \log \log N$</td>
</tr>
<tr>
<td>2007</td>
<td>Fürer</td>
<td>$N \log N 2^{\log^*N}$</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
<td>$N$</td>
</tr>
</tbody>
</table>

number of bit operations to multiply two $N$–bit integers

Remark. GNU Multiple Precision Library uses one of five different algorithm depending on size of operands.
**Numerical linear algebra reductions**

**Matrix multiplication.** Given two $N$-by-$N$ matrices, compute their product. **Brute force.** $N^3$ flops.

\[
\begin{array}{cc}
0.1 & 0.2 & 0.8 & 0.1 \\
0.5 & 0.3 & 0.9 & 0.6 \\
0.1 & 0.0 & 0.7 & 0.4 \\
0.0 & 0.3 & 0.3 & 0.1 \\
\end{array}
\times
\begin{array}{cccc}
0.4 & 0.3 & 0.1 & 0.1 \\
0.2 & 0.2 & 0.0 & 0.6 \\
0.0 & 0.0 & 0.4 & 0.5 \\
0.8 & 0.4 & 0.1 & 0.9 \\
\end{array}
= \
\begin{array}{cccc}
0.16 & 0.11 & 0.34 & 0.62 \\
0.74 & 0.45 & 0.47 & 1.22 \\
0.36 & 0.19 & 0.33 & 0.72 \\
0.14 & 0.10 & 0.13 & 0.42 \\
\end{array}
\]

\[
0.5 \cdot 0.1 + 0.3 \cdot 0.0 + 0.9 \cdot 0.4 + 0.6 \cdot 0.1 = 0.47
\]
Numerical linear algebra reductions


<table>
<thead>
<tr>
<th>problem</th>
<th>linear algebra</th>
<th>order of growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>matrix multiplication</td>
<td>$A \times B$</td>
<td>$MM(N)$</td>
</tr>
<tr>
<td>matrix inversion</td>
<td>$A^{-1}$</td>
<td>$MM(N)$</td>
</tr>
<tr>
<td>determinant</td>
<td>$</td>
<td>A</td>
</tr>
<tr>
<td>system of linear equations</td>
<td>$Ax = b$</td>
<td>$MM(N)$</td>
</tr>
<tr>
<td>LU decomposition</td>
<td>$A = LU$</td>
<td>$MM(N)$</td>
</tr>
<tr>
<td>least squares</td>
<td>$\min |Ax - b|_2$</td>
<td>$MM(N)$</td>
</tr>
</tbody>
</table>

numerical linear algebra problems with the same complexity as matrix multiplication

Q. Is brute-force algorithm optimal?
### History of complexity of matrix multiplication

<table>
<thead>
<tr>
<th>year</th>
<th>algorithm</th>
<th>order of growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>?</td>
<td>brute force</td>
<td>$N^3$</td>
</tr>
<tr>
<td>1969</td>
<td>Strassen</td>
<td>$N^{2.808}$</td>
</tr>
<tr>
<td>1978</td>
<td>Pan</td>
<td>$N^{2.796}$</td>
</tr>
<tr>
<td>1979</td>
<td>Bini</td>
<td>$N^{2.780}$</td>
</tr>
<tr>
<td>1981</td>
<td>Schönhage</td>
<td>$N^{2.522}$</td>
</tr>
<tr>
<td>1982</td>
<td>Romani</td>
<td>$N^{2.517}$</td>
</tr>
<tr>
<td>1982</td>
<td>Coppersmith–Winograd</td>
<td>$N^{2.496}$</td>
</tr>
<tr>
<td>1986</td>
<td>Strassen</td>
<td>$N^{2.479}$</td>
</tr>
<tr>
<td>1989</td>
<td>Coppersmith–Winograd</td>
<td>$N^{2.376}$</td>
</tr>
<tr>
<td>2010</td>
<td>Strother</td>
<td>$N^{2.3737}$</td>
</tr>
<tr>
<td>2012</td>
<td>Williams</td>
<td>$N^{2.372873}$</td>
</tr>
<tr>
<td>2014</td>
<td>de Gall</td>
<td>$N^{2.372864}$</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
<td>$N^{2 + \varepsilon}$</td>
</tr>
</tbody>
</table>

The number of floating-point operations to multiply two $N$-by-$N$ matrices.
6.5 Reductions

- introduction
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- intractability
Bird's-eye view

**Def.** A problem is **intractable** if it can't be solved in polynomial time.

**Desiderata.** Prove that a problem is intractable.

---

**Two problems that provably require exponential time.**

- Given a constant-size program, does it halt in at most \( K \) steps?
- Given \( N \)-by-\( N \) checkers board position, can the first player force a win?

---

**Frustrating news.** Very few successes.
A core problem: satisfiability

**SAT.** Given a system of boolean equations, find a solution.

**Ex.**

\[
\begin{align*}
\neg x_1 \lor x_2 \lor x_3 & = \text{true} \\
x_1 \lor \neg x_2 \lor x_3 & = \text{true} \\
\neg x_1 \lor \neg x_2 \lor \neg x_3 & = \text{true} \\
\neg x_1 \lor \neg x_2 \lor \text{or} \quad \text{or} \quad x_4 & = \text{true} \\
\neg x_2 \lor x_3 \lor x_4 & = \text{true} \\
\end{align*}
\]

<table>
<thead>
<tr>
<th>x_1</th>
<th>x_2</th>
<th>x_3</th>
<th>x_4</th>
<th>solution S</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td></td>
</tr>
</tbody>
</table>

**3-SAT.** All equations of this form (with three variables per equation).

**Key applications.**

- Automatic verification systems for software.
- Mean field diluted spin glass model in physics.
- Electronic design automation (EDA) for hardware.
- ...


Satisfiability is conjectured to be intractable

Q. How to solve an instance of 3-SAT with $N$ variables?
A. Exhaustive search: try all $2^N$ truth assignments.

Q. Can we do anything substantially more clever?

Conjecture ($P \neq NP$). 3-SAT is intractable (no poly-time algorithm).
Polynomial-time reductions

Problem \( X \) poly-time (Cook) reduces to problem \( Y \) if \( X \) can be solved with:

- Polynomial number of standard computational steps.
- Polynomial number of calls to \( Y \).

Establish intractability. If \( 3\text{-SAT} \) poly-time reduces to \( Y \), then \( Y \) is intractable.
(assuming \( 3\text{-SAT} \) is intractable)

Mentality.

- If I could solve \( Y \) in poly-time, then I could also solve \( 3\text{-SAT} \) in poly-time.
- \( 3\text{-SAT} \) is believed to be intractable.
- Therefore, so is \( Y \).
**Integer linear programming**

**ILP.** Given a system of linear inequalities, find an **integral** solution.

\[
3x_1 + 5x_2 + 2x_3 + x_4 + 4x_5 \geq 10 \\
5x_1 + 2x_2 + 4x_4 + 1x_5 \leq 7 \\
x_1 + x_3 + 2x_4 \leq 2 \\
3x_1 + 4x_3 + 7x_4 \leq 7 \\
x_1 + x_4 \leq 1 \\
x_1 + x_3 + x_5 \leq 1
\]

all \( x_i = \{0, 1\} \)

**Context.** Cornerstone problem in operations research.

**Remark.** Finding a real-valued solution is tractable (linear programming).
3-SAT poly-time reduces to ILP

3-SAT. Given a system of boolean equations, find a solution.

\[
\neg x_1 \text{ or } x_2 \text{ or } x_3 = \text{ true} \\
x_1 \text{ or } \neg x_2 \text{ or } x_3 = \text{ true} \\
\neg x_1 \text{ or } \neg x_2 \text{ or } \neg x_3 = \text{ true} \\
\neg x_1 \text{ or } \neg x_2 \text{ or } \neg x_3 \text{ or } x_4 = \text{ true} \\
\neg x_2 \text{ or } x_3 \text{ or } x_4 = \text{ true}
\]

ILP. Given a system of linear inequalities, find a 0-1 solution.

\[
(1 - x_1) + x_2 + x_3 \geq 1 \\
x_1 + (1 - x_2) + x_3 \geq 1 \\
(1 - x_1) + (1 - x_2) + (1 - x_3) \geq 1 \\
(1 - x_1) + (1 - x_2) + x_4 \geq 1 \\
(1 - x_2) + x_3 + x_4 \geq 1
\]

solution to this ILP instance gives solution to original 3-SAT instance
Suppose that Problem $X$ poly-time reduces to Problem $Y$. Which of the following can you infer?

A. If $X$ can be solved in poly-time, then so can $Y$.
B. If $X$ cannot be solved in cubic time, $Y$ cannot be solved in poly-time.
C. If $Y$ can be solved in cubic time, then $X$ can be solved in poly-time.
D. If $Y$ cannot be solved in poly-time, then neither can $X$.
E. I don't know.
More poly-time reductions from 3-satisfiability

3-SAT

- 3-COLOR
  - 3-COLOR reduces to ILP
    - ILP
      - Exact-Cover
        - Exact-Cover reduces to Subset-Sum
          - Subset-Sum
            - Partition
              - Knapsack
              - Bin-Packing
        - Exact-Cover reduces to Ham-Cycle
          - Ham-Cycle
            - TSP
            - Ham-Path

Vertex-Cover

ILP

Conjecture. 3-Sat is intractable.

Implication. All of these problems are intractable.
Implications of poly-time reductions from 3-satisfiability

Establishing intractability through poly-time reduction is an important tool in guiding algorithm design efforts.

Q. How to convince yourself that a new problem is (probably) intractable?
A1. [hard way] Long futile search for an efficient algorithm (as for 3-SAT).
A2. [easy way] Reduction from 3-SAT.

Caveat. Intricate reductions are common.
Search problems

Search problem. Problem where you can check a solution in poly-time.

Ex 1. 3-SAT.

\[
\begin{align*}
\neg x_1 & \text{ or } x_2 \text{ or } x_3 & = & \text{true} \\
x_1 & \text{ or } \neg x_2 \text{ or } x_3 & = & \text{true} \\
\neg x_1 & \text{ or } \neg x_2 \text{ or } \neg x_3 & = & \text{true} \\
\neg x_1 & \text{ or } \neg x_2 & \text{ or } x_4 & = \text{true} \\
\neg x_2 & \text{ or } x_3 & \text{ or } x_4 & = \text{true}
\end{align*}
\]

instance I

\[
\begin{array}{cccc}
x_1 & x_2 & x_3 & x_4 \\
T & T & F & T
\end{array}
\]
solution S

Ex 2. FACTOR. Given an N-bit integer \( x \), find a nontrivial factor.

\[
147573952589676412927 \quad 193707721
\]

instance I

solution S
P vs. NP

**P.** Set of search problems solvable in poly-time.

*Importance.* What scientists and engineers can compute feasibly.

**NP.** Set of search problems (checkable in poly-time).

*Importance.* What scientists and engineers aspire to compute feasibly.

Fundamental question.

Consensus opinion. No.
Cook-Levin theorem

A problem is **NP–COMPLETE** if

- It is in **NP**.
- All problems in **NP** poly-time reduce to it.

**Cook-Levin theorem.** 3-SAT is **NP–COMPLETE**.

**Corollary.** 3-SAT is tractable if and only if \( P = NP \).

Two worlds.

\[
\begin{array}{c}
\text{NP} \\
\text{P} & \text{NPC} \\
P \neq \text{NP}
\end{array}
\quad
\begin{array}{c}
P = \text{NP} \\
P = \text{NP}
\end{array}
\]
Implications of Cook-Levin theorem

All of these problems (and many, many more) poly-time reduce to 3-SAT.
Implications of Karp + Cook-Levin

All of these problems are NP-COMPLETE; they are manifestations of the same really hard problem.
Reductions: quiz 4

Suppose that $X$ is **NP-COMPLETE**, $Y$ is in **NP**, and $X$ poly-time reduces to $Y$. Which of the following statements can you infer?

I. $Y$ is **NP-COMPLETE**.
II. If $Y$ cannot be solved in poly-time, then $P \neq NP$.
III. If $P \neq NP$, then neither $X$ nor $Y$ can be solved in poly-time.

A. I only.
B. II only.
C. I and II only.
D. I, II, and III.
E. $I$ don't know.
**Birds-eye view: review**

**Desiderata.** Classify problems according to computational requirements.

<table>
<thead>
<tr>
<th>complexity</th>
<th>order of growth</th>
<th>examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear</td>
<td>$N$</td>
<td><em>min, max, median, Burrows-Wheeler transform, ...</em></td>
</tr>
<tr>
<td>linearithmic</td>
<td>$N \log N$</td>
<td><em>sorting, element distinctness, ...</em></td>
</tr>
<tr>
<td>quadratic</td>
<td>$N^2$</td>
<td>?</td>
</tr>
<tr>
<td></td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>exponential</td>
<td>$c^N$</td>
<td>?</td>
</tr>
</tbody>
</table>

**Frustrating news.** Huge number of problems have defied classification.
**Desiderata.** Classify *problems* according to computational requirements.

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<td>( N \log N )</td>
<td><em>sorting, element distinctness, ...</em></td>
</tr>
<tr>
<td>M(N)</td>
<td>?</td>
<td><em>integer multiplication, division, square root, ...</em></td>
</tr>
<tr>
<td>MM(N)</td>
<td>?</td>
<td><em>matrix multiplication, ( Ax = b ), least square, determinant, ...</em></td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>NP-complete</td>
<td><em>probably not ( N^b )</em></td>
<td><em>3-SAT, IND-SET, ILP, ...</em></td>
</tr>
</tbody>
</table>

**Good news.** Can put many problems into equivalence classes.
Complexity zoo

Complexity class. Set of problems sharing some computational property.

Bad news. Lots of complexity classes (498 animals in zoo).
Summary

Reductions are important in theory to:

- Design algorithms.
- Establish lower bounds.
- Classify problems according to their computational requirements.

Reductions are important in practice to:

- Design algorithms.
- Design reusable software modules.
  - stacks, queues, priority queues, symbol tables, sets, graphs
  - sorting, regular expressions, suffix arrays
  - MST, shortest paths, maxflow, linear programming
- Determine difficulty of your problem and choose the right tool.