Laplacian Meshes

COS 526 – Fall 2014
Slides from Olga Sorkine and Yaron Lipman
Outline

• Differential surface representation
• Ideas and applications
  – Compact shape representation
  – Mesh editing and manipulation
  – Membrane and flattening
  – Generalizing Fourier basis for surfaces
Motivation

• Meshes are great, but:
  – Geometry is represented in a *global* coordinate system
    • Single Cartesian coordinate of a vertex doesn’t say much
Laplacian Mesh Editing

- Meshes are difficult to edit
Motivation

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- Meshes are difficult to edit
Differential coordinates

- Represent a point *relative* to its neighbors.
- Represent *local detail* at each surface point
  – better describe the shape
- Linear transition from global to differential
- Useful for operations on surfaces where surface details are important
Differential coordinates

“Local control for mesh morphing”, Alexa 01

- Detail = surface – smooth(surface)
- Smoothing = averaging

\[ \delta_i = v_i - \frac{1}{d_i} \sum_{j \in N(i)} v_j \]

\[ \delta_i = \sum_{j \in N(i)} \frac{1}{d_i} (v_i - v_j) \]
Connection to the smooth case

- The direction of \( \delta_i \) approximates the normal
- The size approximates the mean curvature

\[
\delta_i = \frac{1}{d_i} \sum_{v \in N(i)} (v_i - v)
\]

\[
\frac{1}{\text{len}(\gamma)} \int_{v \in \gamma} (v_i - v) \, ds
\]

\[
\lim_{\text{len}(\gamma) \to 0} \frac{1}{\text{len}(\gamma)} \int_{v \in \gamma} (v_i - v) \, ds = H(v_i) n_i
\]
Laplacian matrix

The mesh

The symmetric Laplacian $L_s$
Weighting schemes

\[ \delta_i = \frac{\sum_{j \in N(i)} w_{ij} (v_i - v_j)}{\sum_{j \in N(i)} w_{ij}} \]

- Ignore geometry
  \[ \delta_{\text{umbrella}} : w_{ij} = 1 \]
- Integrate over circle around vertex
  \[ \delta_{\text{mean value}} : w_{ij} = \tan \phi_{ij}/2 + \tan \phi_{ij+1}/2 \]
- Integrate over Voronoi region of vertex
  \[ \delta_{\text{cotangent}} : w_{ij} = \cot \alpha_{ij} + \cot \beta_{ij} \]
Laplacian mesh

- Vertex positions are represented by Laplacian coordinates \((\delta_x, \delta_y, \delta_z)\)

\[
\delta_i = \sum_{j \in N(i)} w_{ij} (v_i - v_j)
\]

\[
L \cdot v_x = \delta_x
\]

\[
L \cdot v_y = \delta_y
\]

\[
L \cdot v_z = \delta_z
\]
Basic properties

- \( \text{rank}(L) = n - c \) (\( n - 1 \) for connected meshes)
- We can reconstruct the \( xyz \) geometry from \( \delta \) up to translation
Reconstruction

\[ c_x \times 1 = L \]
\[ v_x = \delta_x \]
\[ c_y \times 1 = L \]
\[ v_y = \delta_y \]
\[ c_z \times 1 = L \]
\[ v_z = \delta_z \]
Reconstruction

\[ \tilde{x} = \arg \min_x \left( \| Lx - \delta_x \|^2 + \sum_{s=1}^{k} |x_k - c_k|^2 \right) \]
Reconstruction

\[
L v_x = \delta_x
\]

\[
A x = b
\]

Normal Equations:

\[
A^T A x = A^T b
\]

\[
x = (A^T A)^{-1} A^T b
\]

compute once
Cool underlying idea

- Mesh vertex positions are defined by minimizer of an objective function

$$\tilde{x} = \arg\min_x \left( \|Lx - \delta_x\|^2 + \sum_{s=1}^k |x_k - c_k|^2 \right)$$
What we have so far

- Laplacian coordinates $\delta$
  - Local representation
  - Translation-invariant

- Linear transition from $\delta$ to $xyz$
  - can constrain more than 1 vertex
  - least-squares solution
Editing using differential coordinates

• The editing process from the user’s point of view:
  1) First, a ROI, anchors and a handle vertex should be set.
  2) Then the edit is performed by moving this vertex.
Editing using differential coordinates

- The user moves the handle and **interactively** the surface changes.
- The stationary anchors are responsible for **smooth transition** of the edited part to the rest of the mesh.
- This is done using increasing weight with geodesic distance in the **soft** spatial equations.
Mesh Editing Example
Mesh Editing Example
Mesh Editing Example
Mesh Editing Example
What else can we do with it?
Parameterization

- Use zero Laplacians.

In 2D:
Texture Mapping
Texture Mapping

[Pipeoni2000]
Feature Preserving Smoothing

- Weighted positional and smoothing constraints

\[
\begin{align*}
L & = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \times V \\
& = P \\
& = \begin{bmatrix} 0 \end{bmatrix}
\end{align*}
\]
Feature Preserving Smoothing

- Weighted positional and smoothing constraints
Detail transfer

• “Peel“ the coating of one surface and transfer to another
Detail transfer
Detail transfer
Mixing Laplacians

- Taking weighted average of $\delta_i$ and $\delta'_i$
Mesh transplanting

- Geometrical stitching via Laplacian mixing
Mesh transplanting

- Details gradually change in the transition area
Mesh transplanting

- Details gradually change in the transition area
The End