Matching & Assignment Problems

N residents    N hospitals

NRMP

Formulating as an algorithm

Input: \rightarrow \text{for each resident } r_i \text{ \ a list of hospitals (a relation } >_{r_i} \text{ \ on the set of hospitals)}

\begin{align*}
& h_1 >_{r_i} h_2 \quad \text{if } r_i \text{ prefers } h_1 \text{ over } h_2 \\
\rightarrow \text{for each hospital } h_j \text{ \ a ranked list of residents} \\
& h_j >_{h_j} h_j
\end{align*}

Output: Matching

What do we want from the matching?

Need people to listen (abstractly)

\rightarrow \text{stability condition.}

Let Matching be \( M(r_i) = h_j \)

\begin{align*}
& r_1 \leftarrow M(r_1) \quad \text{not stable:} \\
& r_2 \leftarrow M(r_2) \quad M(r_2) >_{r_1} r_1 \\
& r_2 \leftarrow M(r_2) \\& M(r_2) \leftarrow r_1
\end{align*}
A pair $r_1, M(r_2)$

S.t. $M(r_1) < r_1 M(r_2)$

$r_2 < M(r_2) r_1$

is called a **blocking pair**

A match is stable if there is no blocking pair

Defended Acceptance Algorithm (DA) Gale-Shapley

**At Repeat:**

1. Each resident applies down their list.
2. If a hospital receives multiple applications, it rejects all but its top choice.

**Until**: All residents are assigned or have exhausted their list

**Example:**

\[
\begin{array}{ccc}
\gamma_1, \gamma_2, \gamma_3 & A & B & C \\
A & A & B & r_3 & r_2 & r_1 \\
C & B & A & r_1 & r_3 & r_2 \\
B & C & C & r_2 & r_1 & r_3 \\
\end{array}
\]
Execution

\[
\begin{array}{ccc}
A & \alpha & C \\
\gamma_1 & \gamma_2 & \gamma_3 \\
\gamma_3 & \gamma_2 & \gamma_1
\end{array}
\]

Claim: DA outputs a stable match.

Proof: We will prove there is no blocking pair $r-h$

Observation: As the algorithm progresses, the hospital's situation only improves.

\[\text{By contradiction}\]

$r-h$ is a blocking pair, then $r$ prefers $h$ to its ultimate outcome

\[\Rightarrow r \text{ had applied to } h\]

\[\Rightarrow \text{By observation, this means that } h \text{'s outcome is preferred by } r \text{ to } r.\]

To complete correctness, notice that DA takes $\leq n^2$ iterations.
Cor: Is stable match, DA finds it

As the stable match unique?

1) Any execution of DA above produces the same matches

Resident - proposing DA

Hospital proposing DA

---

Try with 2

<table>
<thead>
<tr>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
</tr>
</tbody>
</table>

\[
\begin{array}{cc}
\gamma_1 & \gamma_2 \\
A & A \\
B & B \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma_1 & \gamma_2 \\
A & B \\
B & A \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma_1 & \gamma_2 \\
\gamma_2 & \gamma_1 \\
\gamma_1 & \gamma_2 \\
\end{array}
\]

Yes proposing $\gamma_1 \rightarrow A$

DA

Yes proposing $\gamma_2 \rightarrow B$

Two stable matches

hosp. proposing $\gamma_2 \rightarrow A$

DA

hosp. proposing $\gamma_1 \rightarrow B$
Theorem

Resident – proposing – DA results in a pointwise best outcome for residents and pointwise worst for hospitals.

If $M_{\text{RDA}}$ is the matching and $M$ is any stable match:

$M_{\text{RDA}}(r) \succeq r \quad \forall r$

$M_{\text{RDA}}^{-1}(h) \preceq h^{\text{M}} \quad \forall h$

Algorithm to Mechanism

Runs on strings

Runs on inputs from people (will need incentives)
A mechanism is truthful if a player can't improve their outcome by misreporting preferences.

Is RDA truthful?

Yes for residents
No for hospitals

\[
\begin{array}{cccccc}
\gamma_3 & A & B & C & \gamma_1 & \gamma_2 & \gamma_3 \\
\gamma_2 & \gamma_2 & \gamma_3 & A & B & A \\
\gamma_1 & \gamma_2 & \gamma_1 & B & A & C \\
\gamma_3 & \gamma_3 & \gamma_2 & C & C & C & B \\
\end{array}
\]

Normal

Now A switches $\gamma_1$ and $\gamma_3$
1970 + Couple constraints

\[ r_1 - r_2 \]

individually

\[ c_1 \quad n_1 \]
\[ n_1 \quad c_1 \]
\[ c_2 \quad c_2 \]

There might not be a stable match.

\[
\begin{array}{ccc}
W - M & S & A - B \\
A - B & A & W - S \\
& B & S - m \\
\end{array}
\]

1. (A - B) - S
   \[
   \begin{array}{c}
   W - M \\
   \end{array}
   \]

2. (A - B) - S
   \[
   \begin{array}{c}
   W - m \\
   A - B \\
   S - Q \\
   \end{array}
   \]

3. (A - B) - S
   \[
   \begin{array}{c}
   A - B \\
   S - Q \\
   \end{array}
   \]

No stable match!!
“Large market hypothesis”

Mechanism design without money

→ School matches: grade-schools
Schools might not have preferences. Only students have preferences

Lotteries

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>C</td>
</tr>
</tbody>
</table>

output: A distribution of matches

Random Serial Dictatorship:

Pick a random permutation of player \( \pi \)

For \( i = 1 \) to \( n \)

\( \pi(i) \) picks its favourite remaining item
RS D is truthful: why?

When its your turn, you're the dictator.

Conj: Only truthful assignment with some desirable properties.

Example

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>½ A</td>
<td>½ A</td>
<td>½ B</td>
<td>½ B</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>B</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td>¼ C</td>
<td>¼ C</td>
<td>¼ C</td>
<td>¼ C</td>
</tr>
<tr>
<td></td>
<td>¼ D</td>
<td>¼ D</td>
<td>¼ D</td>
<td>¼ D</td>
</tr>
</tbody>
</table>

Ideal mechanism

\[ RS_D \]

D → 1/4
C → 1/4
B → *1/12
A → 5/12