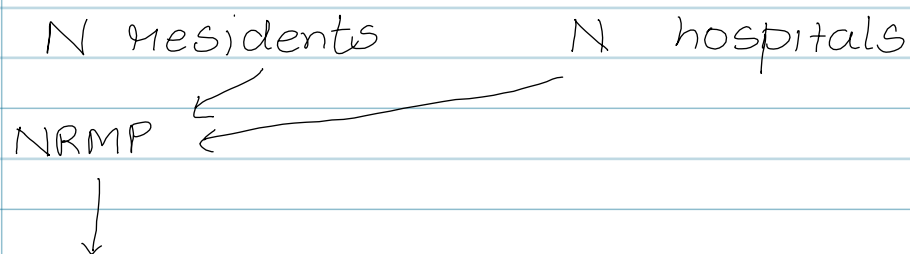


Matching & Assignment Problems



Formulating as an algorithm

Input: → For each resident r_i a list of hospitals (a relation $>_{r_i}$ on the set of hospitals)

$h_1 >_{r_i} h_2$ if r_i prefers h_1 over h_2

→ For each hospital h_j a ranked list of residents $>_{h_j}$

Output: Matching

What do we want from the matching?

Need people to listen (abstractly)

↳ stability condition.

Let Matching be $M(r_i) = h_j$

Not stable:

r_1	=	$M(r_1)$	"	$M(r_2)$	"	r_1
r_2	=	$M(r_2)$	"	$M(r_1)$	"	r_2

r_1 prefers $M(r_2)$ over $M(r_1)$

r_2 prefers $M(r_1)$ over $M(r_2)$

A pair $r_1, M(r_2)$

s.t. $M(r_1) <_{r_1} M(r_2)$

$r_2 <_{M(r_2)} r_1$

is called a blocking pair

A match is stable if there is no blocking pair

Deferred Acceptance Algorithm (DA) Gale-Shapley

At Repeat:

1. Each resident applies down their list.
2. If a hospital receives multiple applications it rejects all but its top choice.

Until: All residents are assigned or have exhausted their list

Example:

r_1, r_2, r_3	A	B	C
A	r_3	r_2	r_1
C	r_1	r_3	r_2
B	r_2	r_1	r_3

Execution

A	B	C
$r_1 r_2$	r_3	r_1
r_3	r_2	

Claim: DA outputs a stable match.

Pf: We will prove there is no blocking pair $r-h$

Observation: As the algorithm progresses, the hospital's situation only improves

pf By contradiction

$r-h$ is a blocking pair, then r

prefers h to its ultimate outcome

$\Rightarrow r$ had \propto applied to h

\Rightarrow By observation, this means that h 's outcome is preferred by it to r .

To complete correctness, notice that DA takes $\leq n^2$ iterations

Cor: \exists stable match, DA finds it

Is the stable match unique?

1) Any execution of DA above produces the same matches

Resident - proposing DA

Hospital proposing DA

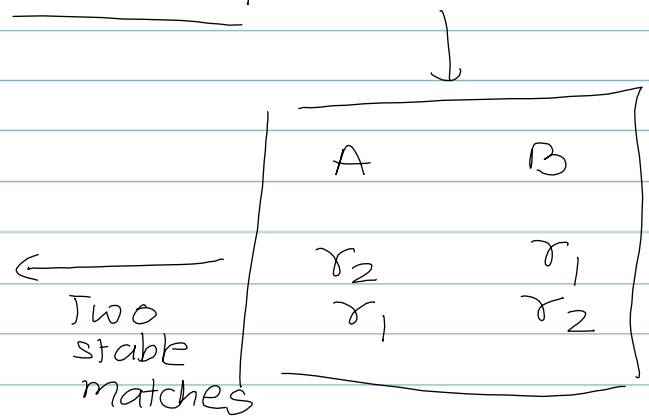
Try with 2

r_1	r_2	
A	A	X
B	B	

r_1	r_2	A	B
A	B	r_1	r_2
B	A	r_2	r_1

res. proposing DA \rightarrow
 $r_1 - A$
 $r_2 - B$

hosp. proposing DA \rightarrow
 $r_2 - A$
 $r_1 - B$



Theorem

Resident - proposing - DA results in a
pointwise best outcome for residents and
pointwise worst for hospitals

If M_{RDA} is the matching and M is any
stable match

$$M_{RDA}(r) \geq_r M(r) \quad \forall r$$

$$M_{RDA}^{-1}(h) \leq_h M^{-1}(h) \quad \forall h$$

Algorithm to Mechanism

↑
runs on strings

↑
runs on inputs from people
(will need incentives)

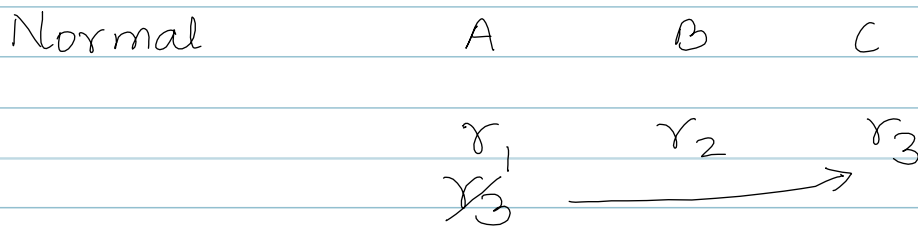
A mechanism is truthful if a player can't improve their outcome by misreporting preferences.

Is rDA truthful?

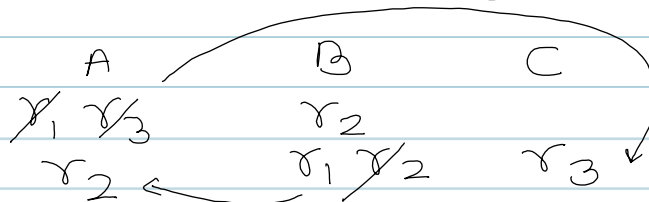
Yes for residents

No for hospitals

r_3	A	B	C	r_1	r_2	r_3
				A	B	A
r_2	r_1	r_3	r_3	B	A	B C
r_1	r_2	r_1		C	C	A B
r_3	r_3	r_2				



Now A switches r_1 and r_3



1970 + Couple Constraints

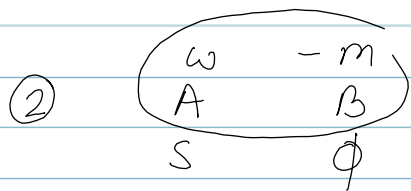
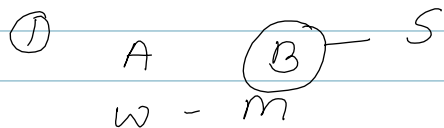
$$r_1 - r_2$$

individually

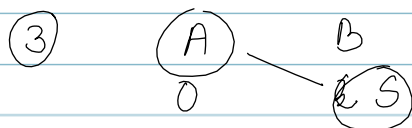
C_1	n_1
n_1	C_1
C_2	C_2

There might not be a stable match ;)

$w - m$	S	A	B
$A - B$	A	w	S
	B	S	m



No stable match !!



"Large market hypothesis"

Mechanism design without money

→ school matches : grade-schools

Schools might not have preferences.
Only students have preferences

Lotteries

input:

1	2	3
A	B	A
B	C	B
C	A	C

output → A distribution of matches

Random Serial Dictatorship :

Pick a random permutation of player
(π)

For $i = 1$ to n

$\pi(i)$ picks its favourite remaining item

RSD is truthful : why?

When its your turn, you're the dictator.

Conj^{??}: Only truthful assignment with some desirable properties.

Example

	1	2	3	4
$\frac{1}{2}$	A	$\frac{1}{2}$ A	$\frac{1}{2}$ B	$\frac{1}{2}$ B
	B	B	A	A
$\frac{1}{4}$	C	$\frac{1}{4}$ C	$\frac{1}{4}$ C	$\frac{1}{4}$ C
$\frac{1}{4}$	D	$\frac{1}{4}$ D	$\frac{1}{4}$ D	$\frac{1}{4}$ D

ideal mechanism

Rsd

D	→	$\frac{1}{4}$
C	→	$\frac{1}{4}$
B	→	$\frac{1}{12}$
A	→	$\frac{5}{12}$