Matching & Assignment Problems

N residents $\qquad$ N hospitals

NRMP

Formulating as an algorithm

Input : → For each resident r_i a list of hospitals (a relation $>_{r_i}$ on the set of hospitals)

$h_1 >_{r_i} h_2$ if r_i prefers h_1 over h_2

→ For each hospital h_J a ranked list of residents

$>_{h_J}$

Output : Matching

What do we want from the matching ?

Need people to listen (abstractly)

↳ Stability condition.

Let Matching be $M(r_i) = h_J$

r_1 ——— $M(r_1)$

r_2 ——— $M(r_2)$

Not stable :

r_1 prefers $M(r_2)$

$M(r_2)$ " r_1

A pair $r_1, M(r_2)$

s.t. $M(r_1) <_{r_1} M(r_2)$

$r_2 <_{M(r_2)} r_1$

is called a <u>blocking pair</u>

A match is <u>stable</u> if there is no blocking pair

<u>Deferred Acceptance Algorithm (DA) Gale-Shapley</u>

~~At~~ <u>Repeat</u>:

1. Each resident applies down their list.
2. If a hospital recieves multiple applications it rejects all but its top choice.

<u>Until</u>: All residents are assigned or have exhausted their list

<u>Example</u>:

r_1	r_2	r_3
A	A	B
C	B	A
B	C	C

A	B	C
r_3	r_2	r_1
r_1	r_3	r_2
r_2	r_1	r_3

Execution

A	B	C
~~r_1~~, ~~r_2~~	~~r_3~~	r_1
r_3	r_2	

Claim: DA outputs a stable match.

Pf: We will prove there is no blocking pair $r-h$

Observation: As the algorithm progresses, the hospital's situation only improves

Pf By contradiction

$r-h$ is a blocking pair, then r prefers h to its ultimate outcome

$\Rightarrow$ r had applied to h

$\Rightarrow$ By obseration, this means that h's outcome is preferred by it to r.

To complete correctness, notice that DA takes $\leq n^2$ iterations

Cor: $\exists$ stable match, DA finds it

Is the stable match unique?

1) Any execution of DA above produces the same matches

Resident - proposing DA

Hospital proposing DA

Try with 2

r_1	r_2
A	A
B	B

×

r_1	r_2	A	B
A	B	r_1	r_2
B	A	r_2	r_1

A	B
r_2	r_1
r_1	r_2

res. proposing DA $\rightarrow$ r_1 – A, r_2 – B

hosp. proposing DA $\rightarrow$ r_2 – A, r_1 – B

$\leftarrow$ Two stable matches

Theorem

Resident - proposing - DA results in a pointwise best outcome for residents and pointwise worst for hospitals

If M_{RDA} is the matching and M is any stable match

$$M_{RDA}(r) \succeq_r M(r) \quad \forall r$$

$$M^{-1}_{RDA}(h) \preceq_h M^{-1}(h) \quad \forall h$$

Algorithm to Mechanism

runs on strings

runs on inputs from people (will need incentives)

A mechanism is <u>truthful</u> if a player cant improve their outcome by misreporting preferences.

Is rDA truthful?

Yes for residents

No for hospitals

r_3

A	B	C	r_1	r_2	r_3
r_2	r_1	r_3	A	B	A
r_1	r_2	r_1	B	A	~~B~~ C
r_3	r_3	r_2	C	C	~~C~~ B

Normal

A	B	C
r_1	r_2	r_3
$\cancel{r_3}$		

Now A switches r_1 and r_3

A	B	C
$\cancel{r_1}$ $\cancel{r_3}$	r_2	
r_2	r_1 $\cancel{r_2}$	r_3

1970 + Couple Constraints

$r_1 - r_2$

individually

c_1	n_1
n_1	c_1
c_2	c_2

There might not be a stable match :(

$w - m$

$A - B$

S
A
B

A	B
w	s
s	m

① A B — S

w - m

② w - m

A B

S ϕ

③ A B

0 — S

No stable match !!

" Large market hypothesis "

Mechanism design without money

→ school matches : grade- schools

Schools might not have preferences.
Only students have preferences

Lotteries

input.

1	2	3
A	B	A
B	C	B
C	A	C

output → A distribution of matches

Random Serial Dictatorship :

Pick a random permutation (π) of player

For $i = 1$ to n

$\pi(i)$ picks its favourite remaining item

RSD is truthful : why?

When its your turn, you're the dictator.

Conj?? : Only truthful assignment with some desirable properties.

Example

1	2	3	4
$\frac{1}{2}$ A	$\frac{1}{2}$ A	$\frac{1}{2}$ B	$\frac{1}{2}$ B
B	B	A	A
$1/4$ C	$1/4$ C	$1/4$ C	$1/4$ C
$1/4$ D	$1/4$ D	$1/4$ D	$1/4$ D

Ideal mechanism

Rsd

D $\rightarrow$ $1/4$

C $\rightarrow$ $1/4$

B $\rightarrow$ $1/12$

A $\rightarrow$ $5/12$