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6.4 MAXIMUM FLOW

- *introduction*
- *Ford-Fulkerson algorithm*
- *maxflow-mincut theorem*
- *analysis of running time*
- *Java implementation*
- *applications*

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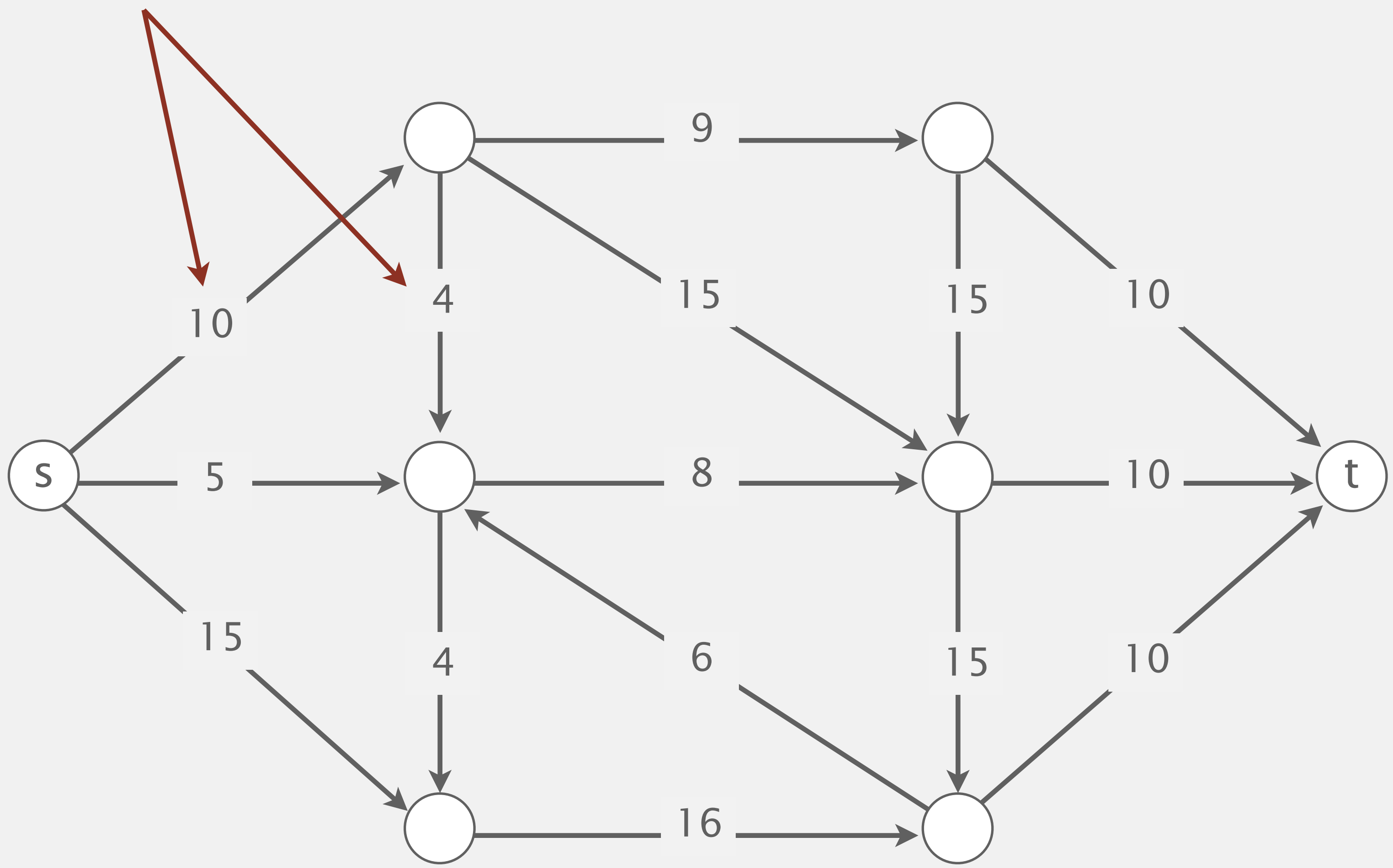
Mincut problem

Input. An edge-weighted digraph, source vertex s, and target vertex t.

each edge has a
positive capacity

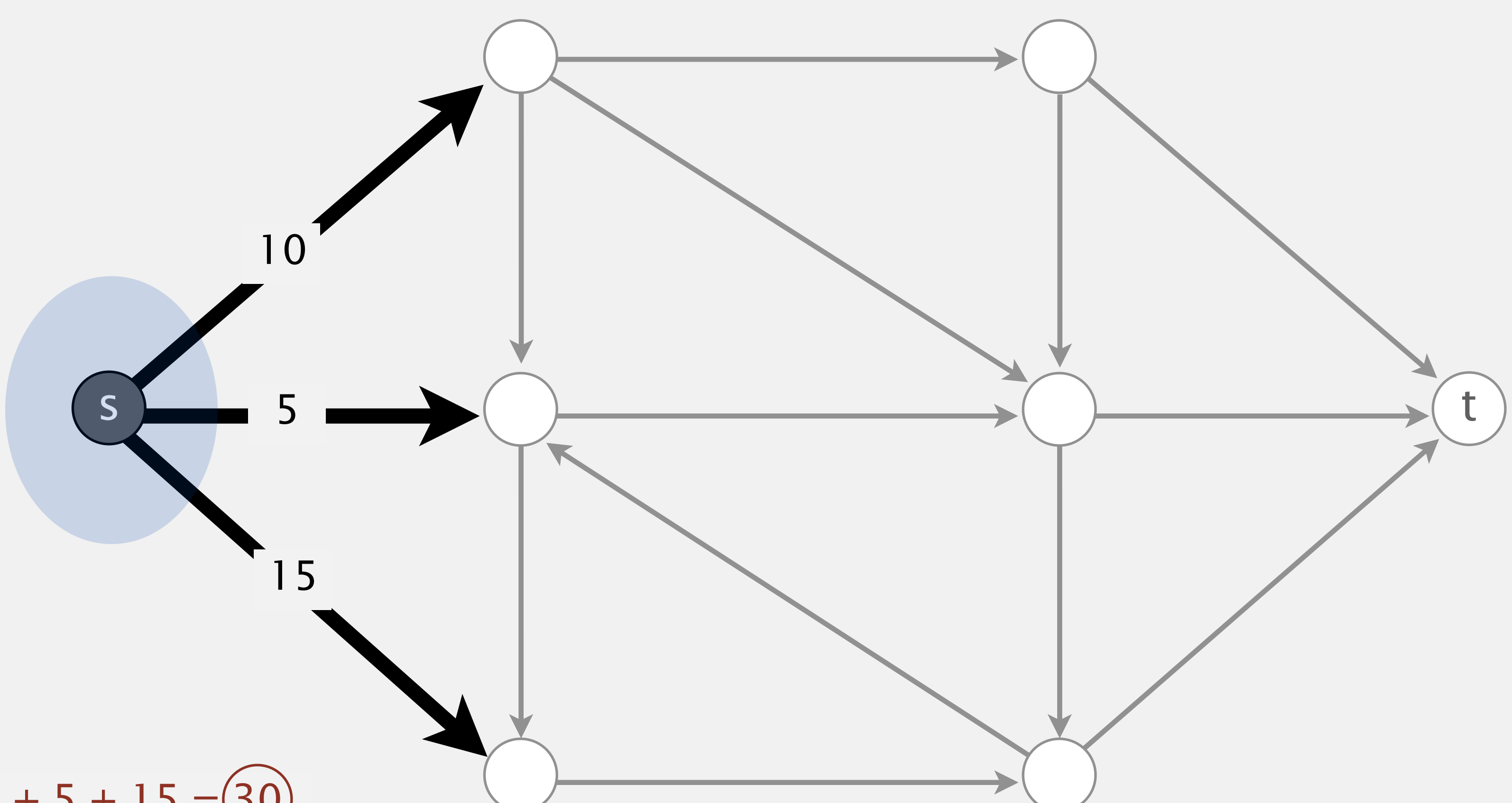
capacity



Mincut problem

Def. A *st*-cut (cut) is a partition of the vertices into two disjoint sets, with s in one set A and t in the other set B.

Def. Its capacity is the sum of the capacities of the edges from A to B.



capacity = 10 + 5 + 15 = 30

Mincut problem

Def. A *st*-cut (cut) is a partition of the vertices into two disjoint sets, with *s* in one set *A* and *t* in the other set *B*.

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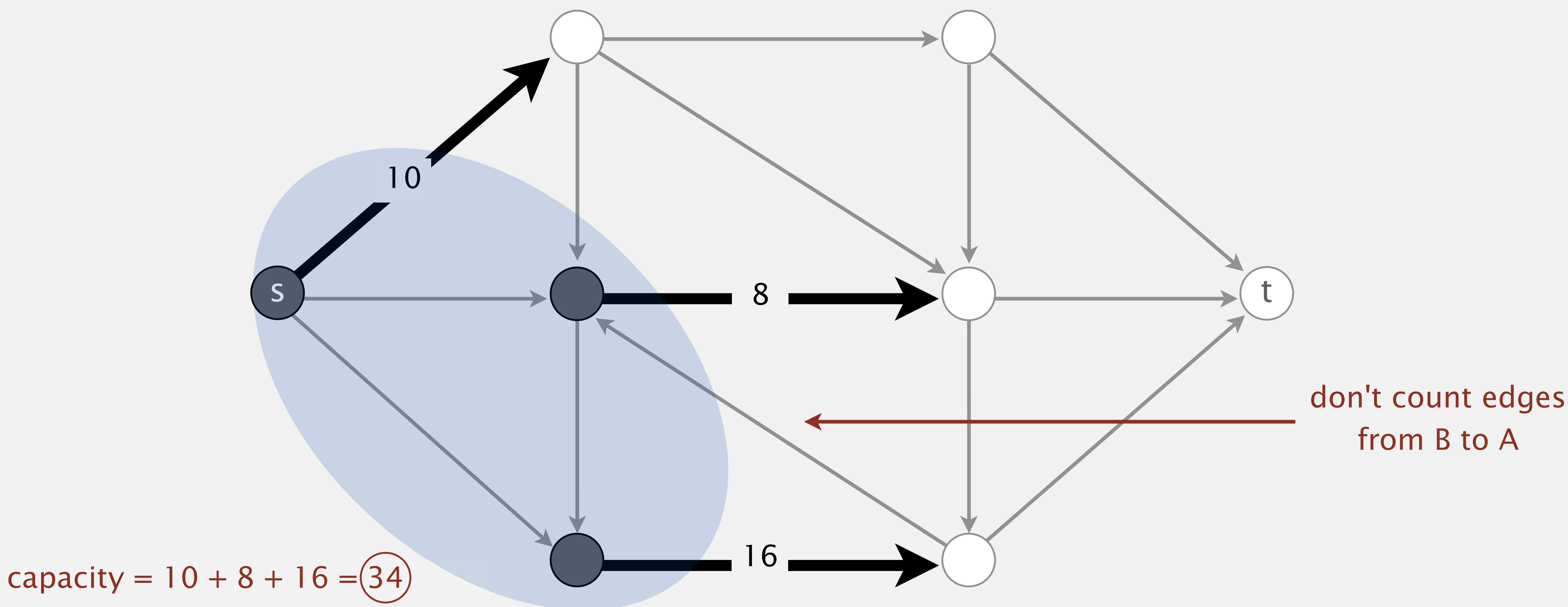
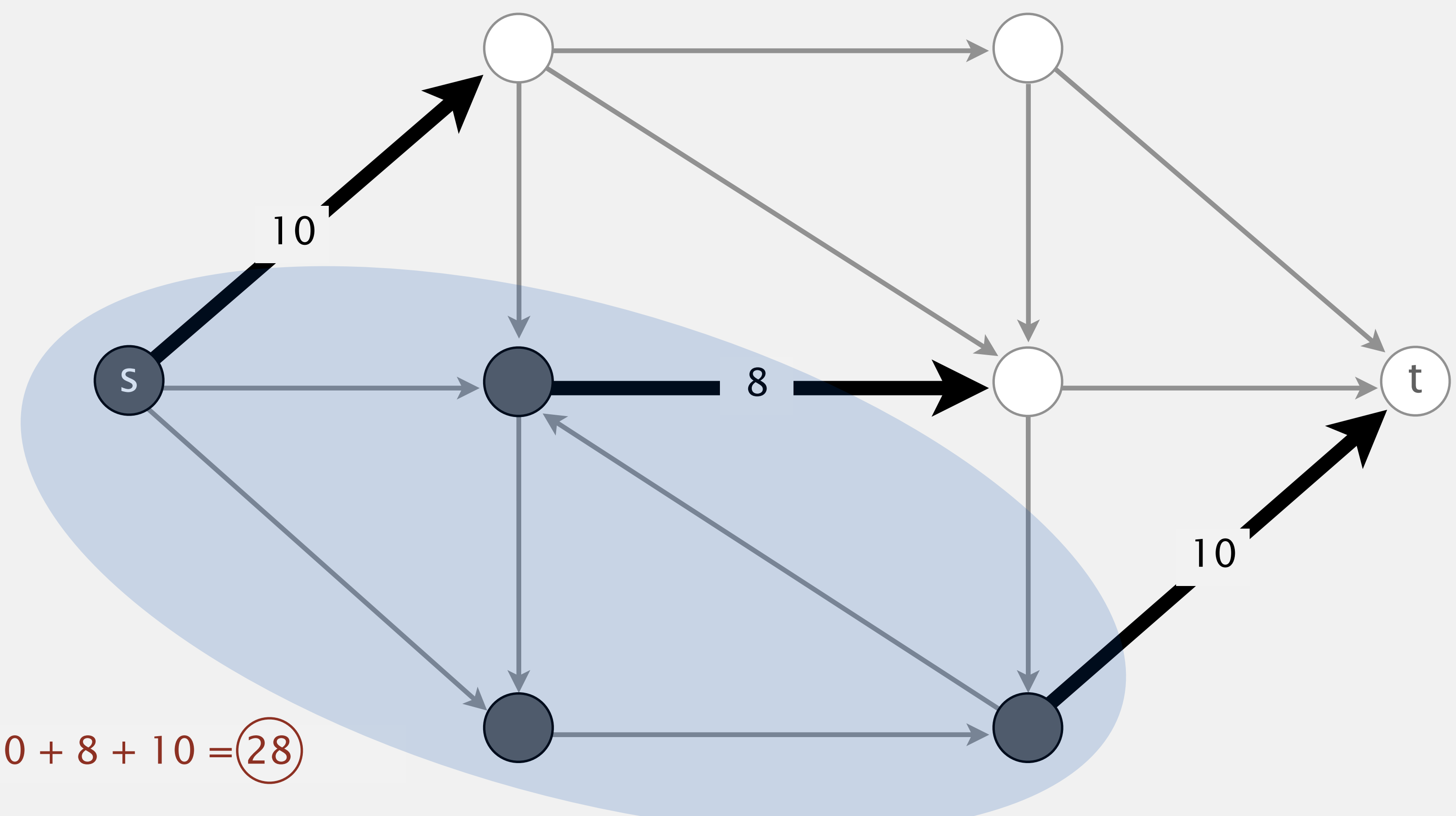


Mincut problem

Def. A *st*-cut (cut) is a partition of the vertices into two disjoint sets, with s in one set A and t in the other set B.

Def. Its capacity is the sum of the capacities of the edges from A to B.

Minimum st-cut (mincut) problem. Find a cut of minimum capacity.



capacity = 10 + 8 + 10 = 28

Maxflow: quiz 1

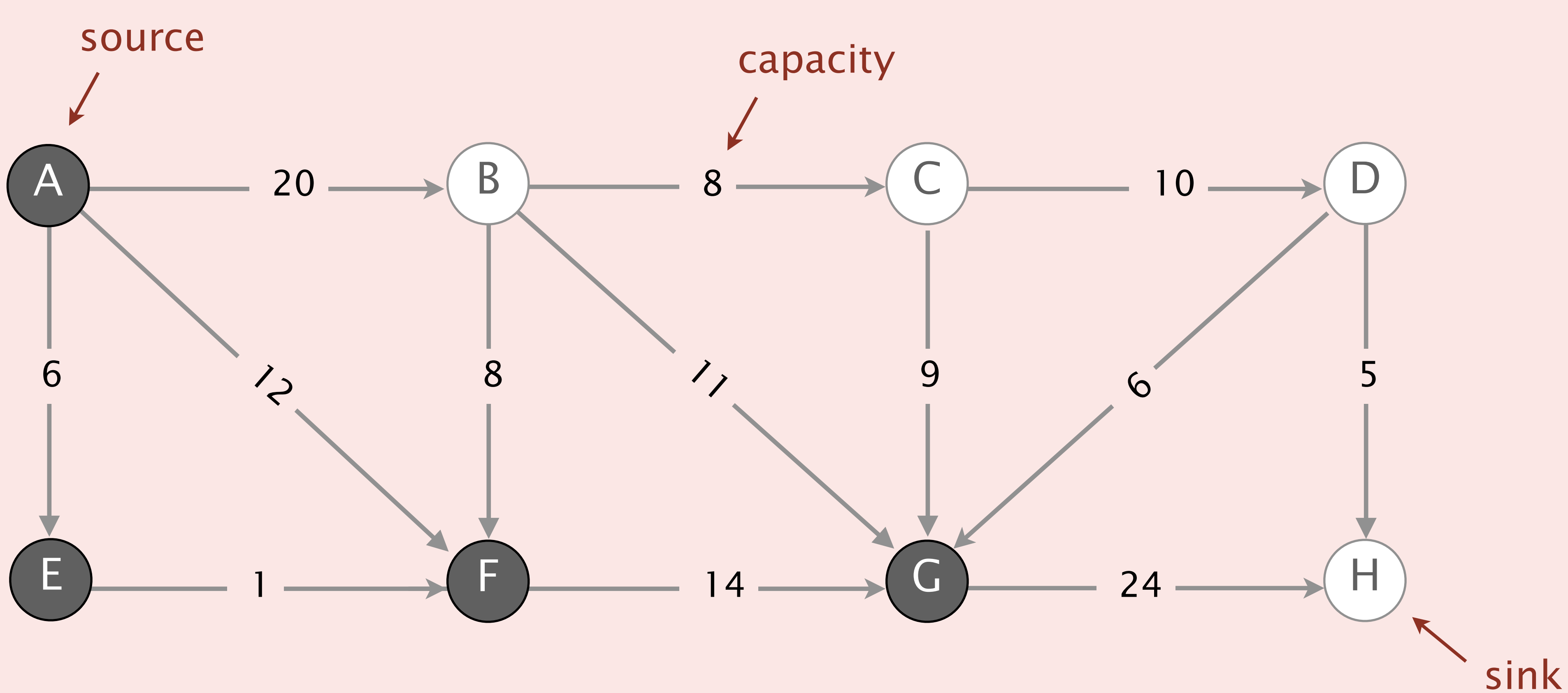
What is the capacity of the st-cut $\{A, E, F, G\}$?

A. 34 (8 + 11 + 9 + 6)

B. 44 (20 + 24)

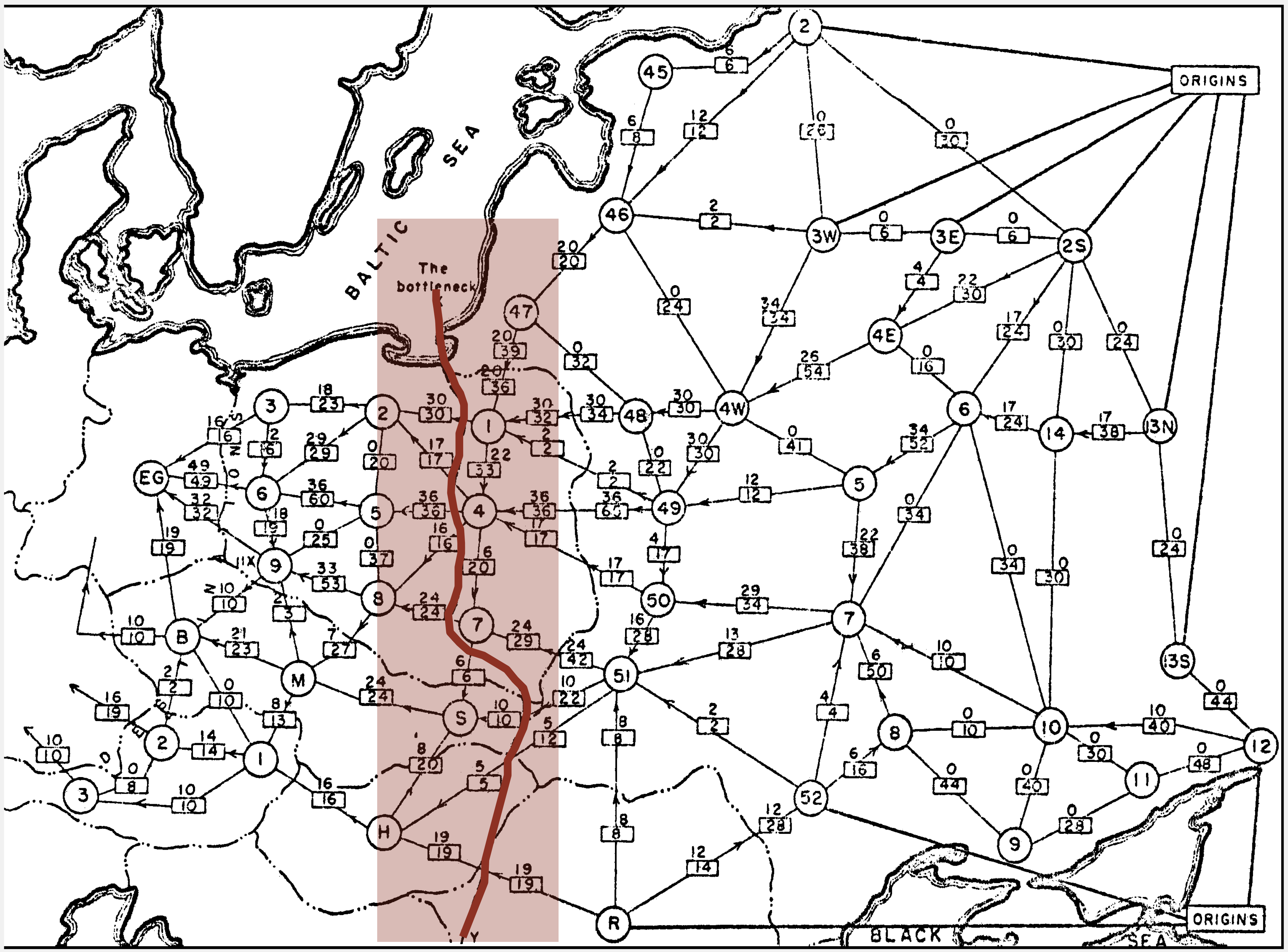
C. 78 (20 + 8 + 11 + 9 + 6 + 24)

D. *I don't know.*



Mincut application (RAND 1950s)

"Free world" goal. Cut supplies (if cold war turns into real war).



rail network connecting Soviet Union with Eastern European countries (map declassified by Pentagon in 1999)

Potential mincut application (2010s)

Government-in-power's goal. Cut off communication to set of people.

Maxflow problem

Though maximum flow algorithms have a long history, revolutionary progress is still being made.

BY ANDREW V. GOLDBERG AND ROBERT E. TARJAN

Efficient Maximum Flow Algorithms

gorithms in more detail. We restrict ourselves to basic maximum flow algorithms and do not cover interesting special cases (such as undirected graphs, planar graphs, and bipartite matchings) or generalizations (such as minimum-cost and multi-commodity flow problems).

Before formally defining the maximum flow and the minimum cut problems, we give a simple example of each problem: For the maximum flow example, suppose we have a graph that represents an oil pipeline network from an oil well to an oil depot. Each arc has a capacity, or maximum number of liters per second that can flow through the corresponding pipe. The goal is to find the maximum number of liters per second (maximum flow) that can be shipped from well to depot. For the minimum cut problem, we want to find the set of pipes of the smallest total capacity such that removing the pipes disconnects the oil well from the oil depot (minimum cut).

The maximum flow, minimum cut

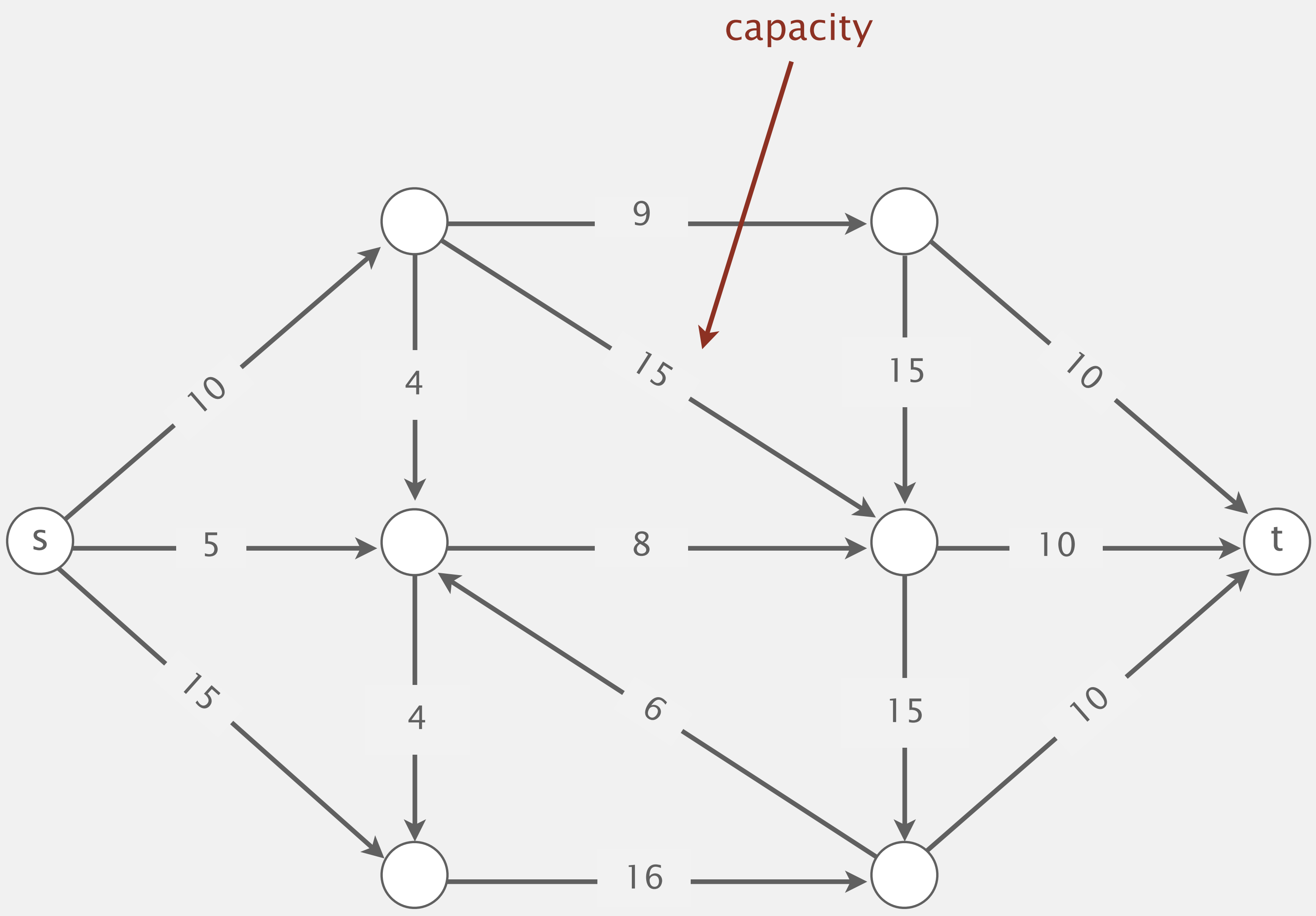
Efficient Maximum Flow Algorithms by Andrew Goldberg and Bob Tarjan

http://vimeo.com/100774435

Maxflow problem

Input. An edge-weighted digraph, source vertex s, and target vertex t.

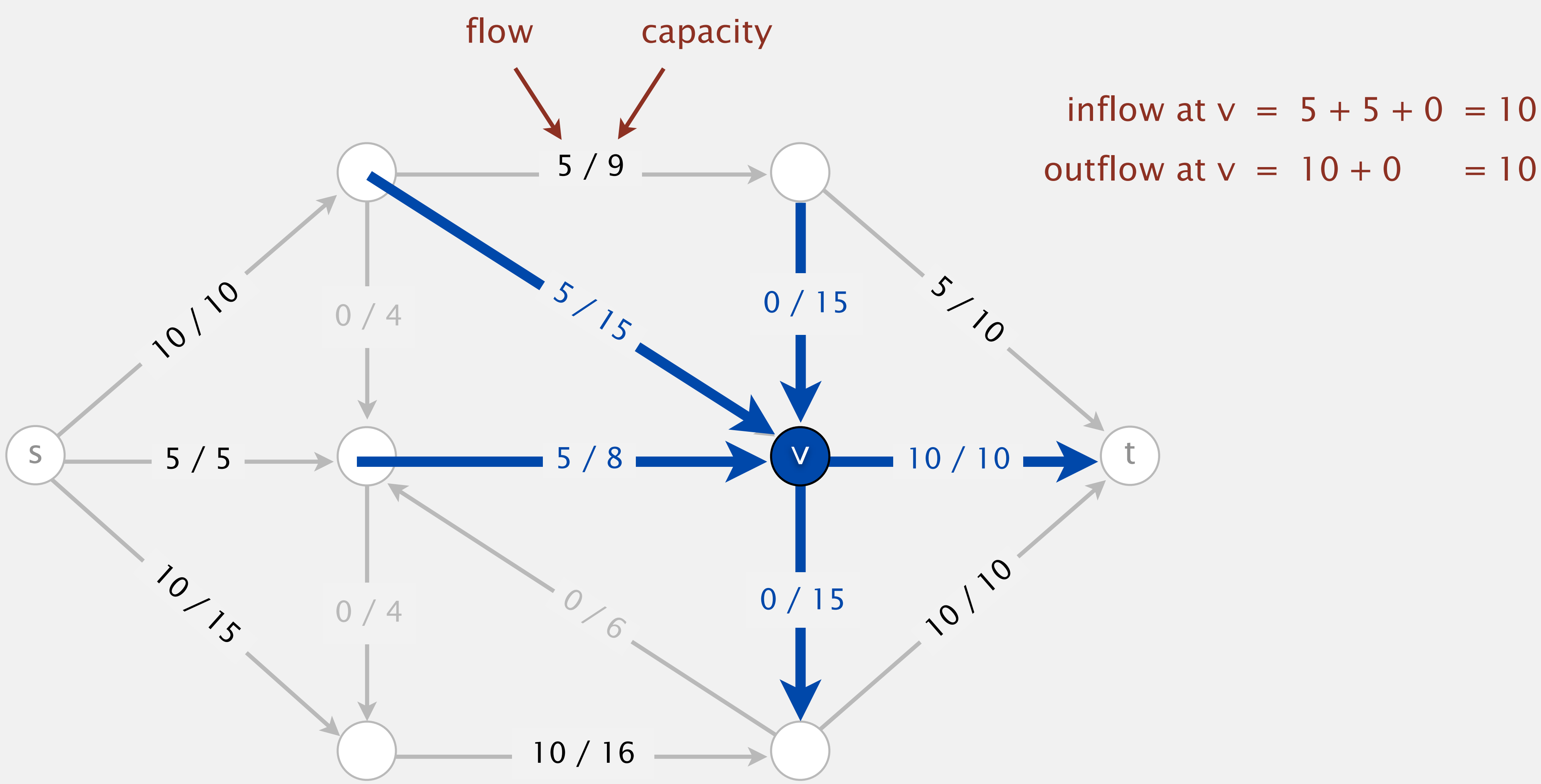
each edge has a
positive capacity



Maxflow problem

Def. An st-flow (flow) is an assignment of values to the edges such that:

- Capacity constraint: $0 \leq$ edge's flow $\leq$ edge's capacity.
- Local equilibrium: inflow = outflow at every vertex (except s and t).



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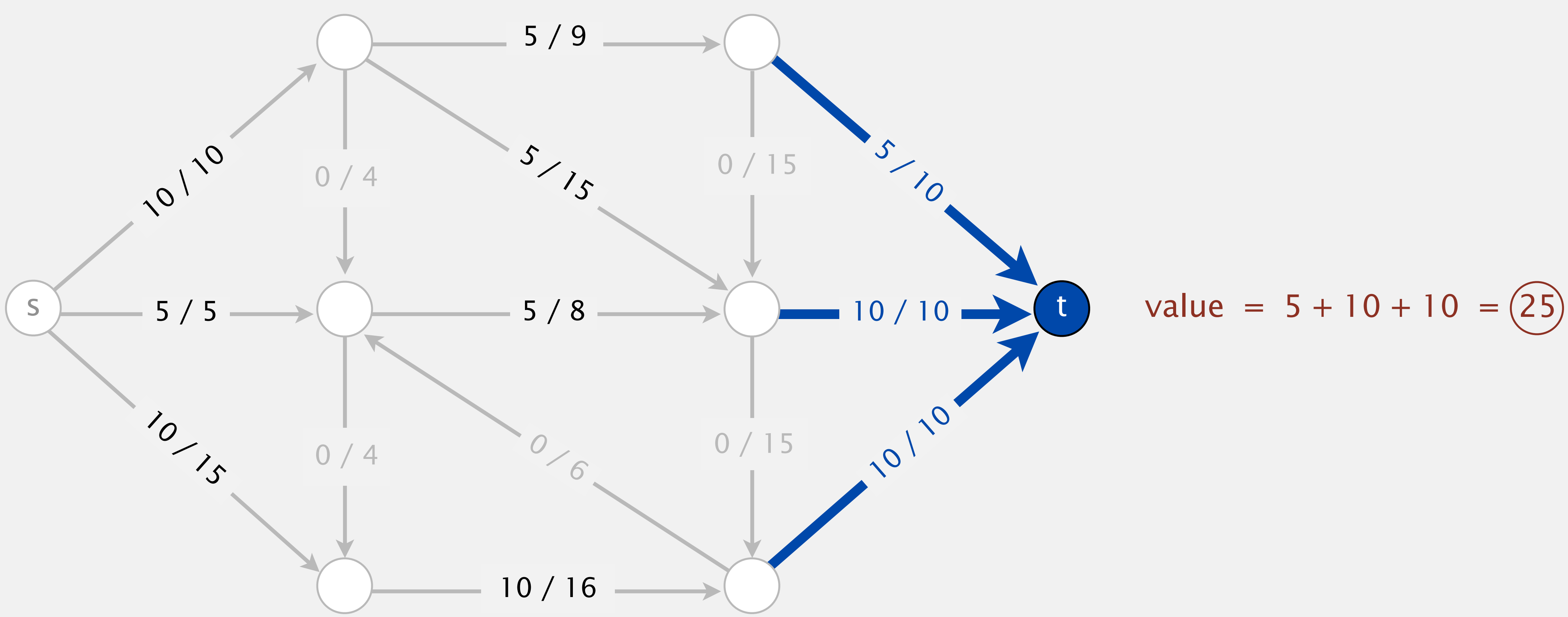
- Capacity constraint: $0 \le$ edge's flow $\le$ edge's capacity.
- Local equilibrium: inflow = outflow at every vertex (except s and t).

Def. The value of a flow is the inflow at t.

we assume no edges point to s or from t



Maxflow problem

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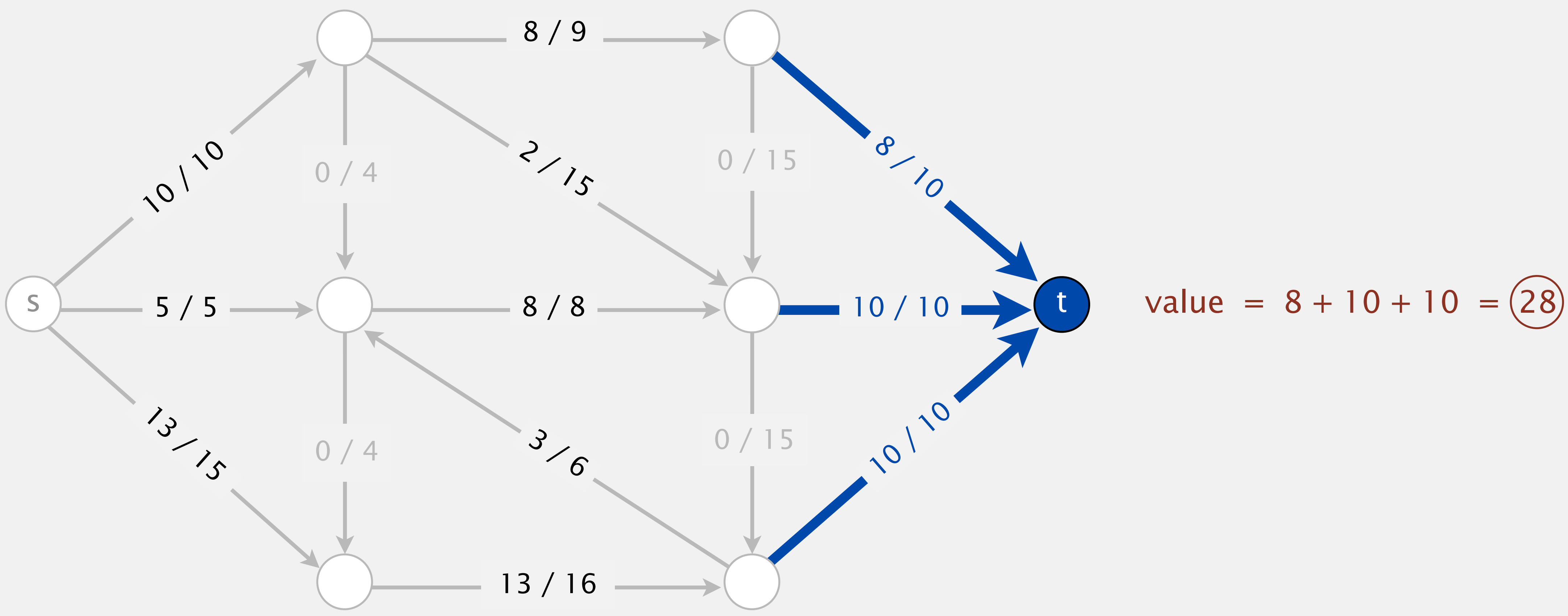
- Capacity constraint: $0 \leq$ edge's flow $\leq$ edge's capacity.
- Local equilibrium: inflow = outflow at every vertex (except s and t).

Def. The value of a flow is the inflow at t.

Maximum st-flow (maxflow) problem. Find a flow of maximum value.

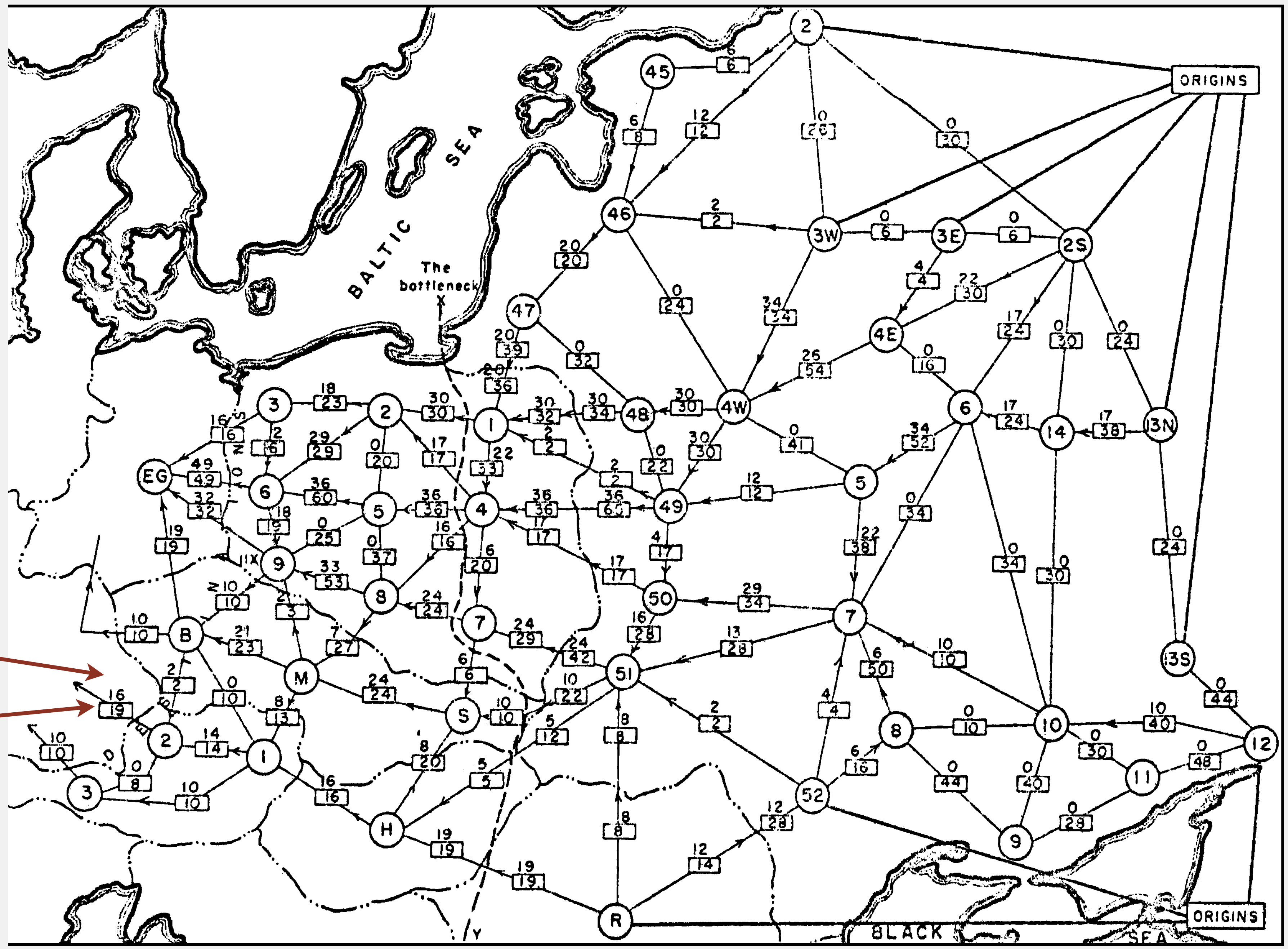


Maxflow application (Tolstoǐ 1930s)

Soviet Union goal. Maximize flow of supplies to Eastern Europe.

flow

capacity



rail network connecting Soviet Union with Eastern European countries

(map declassified by Pentagon in 1999)

Potential maxflow application (2010s)

"Free world" goal. Maximize flow of information to specified set of people.

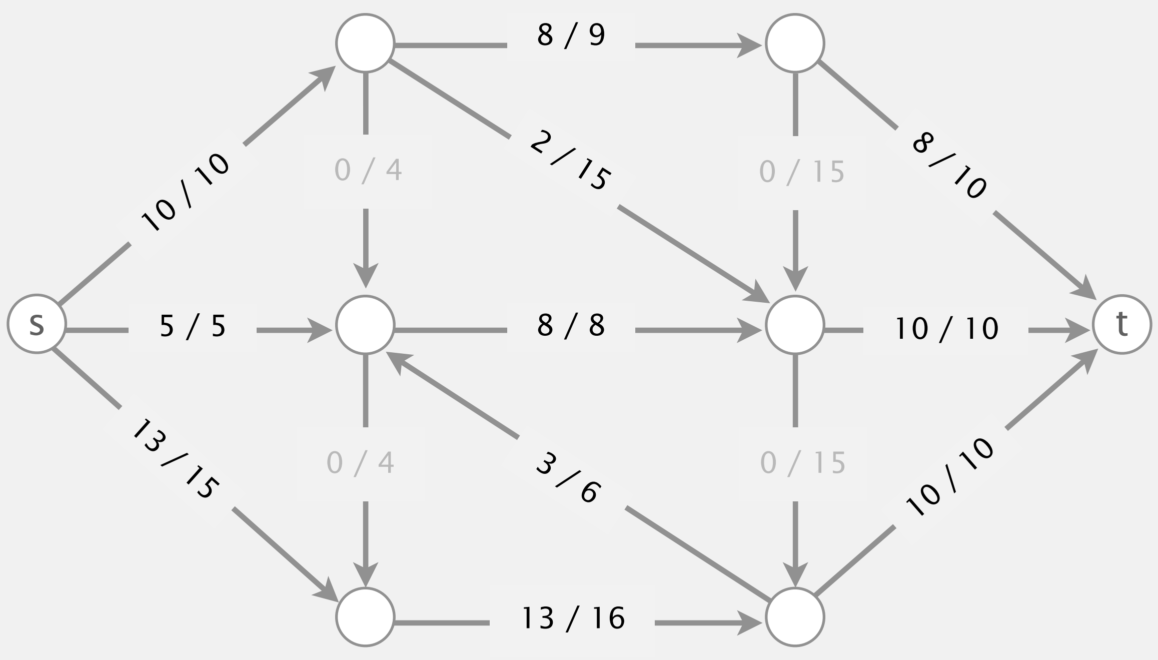
facebook graph

Summary

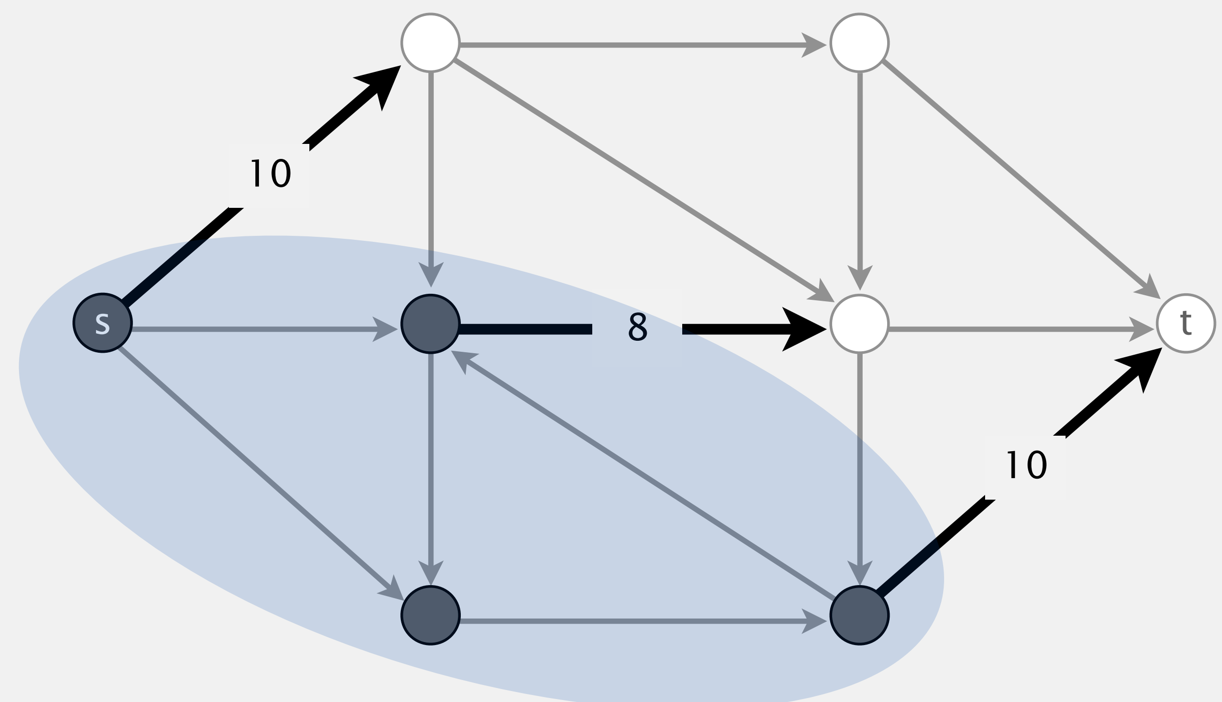
Input. A weighted digraph, source vertex s, and target vertex t.

Mincut problem. Find a cut of minimum capacity.

Maxflow problem. Find a flow of maximum value.



value of flow = 28



capacity of cut = 28

Remarkable fact. These two problems are dual!

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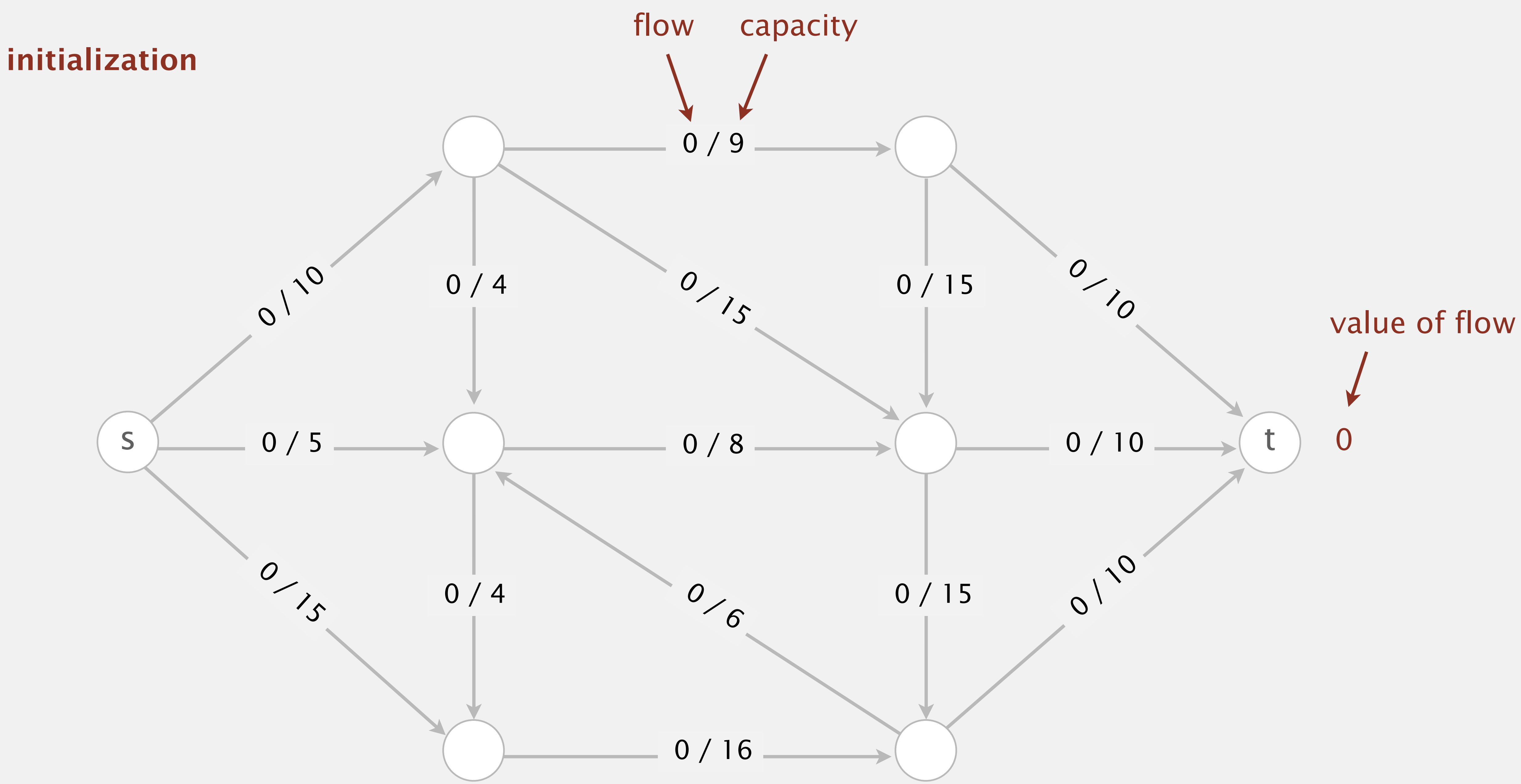
Ford-Fulkerson algorithm

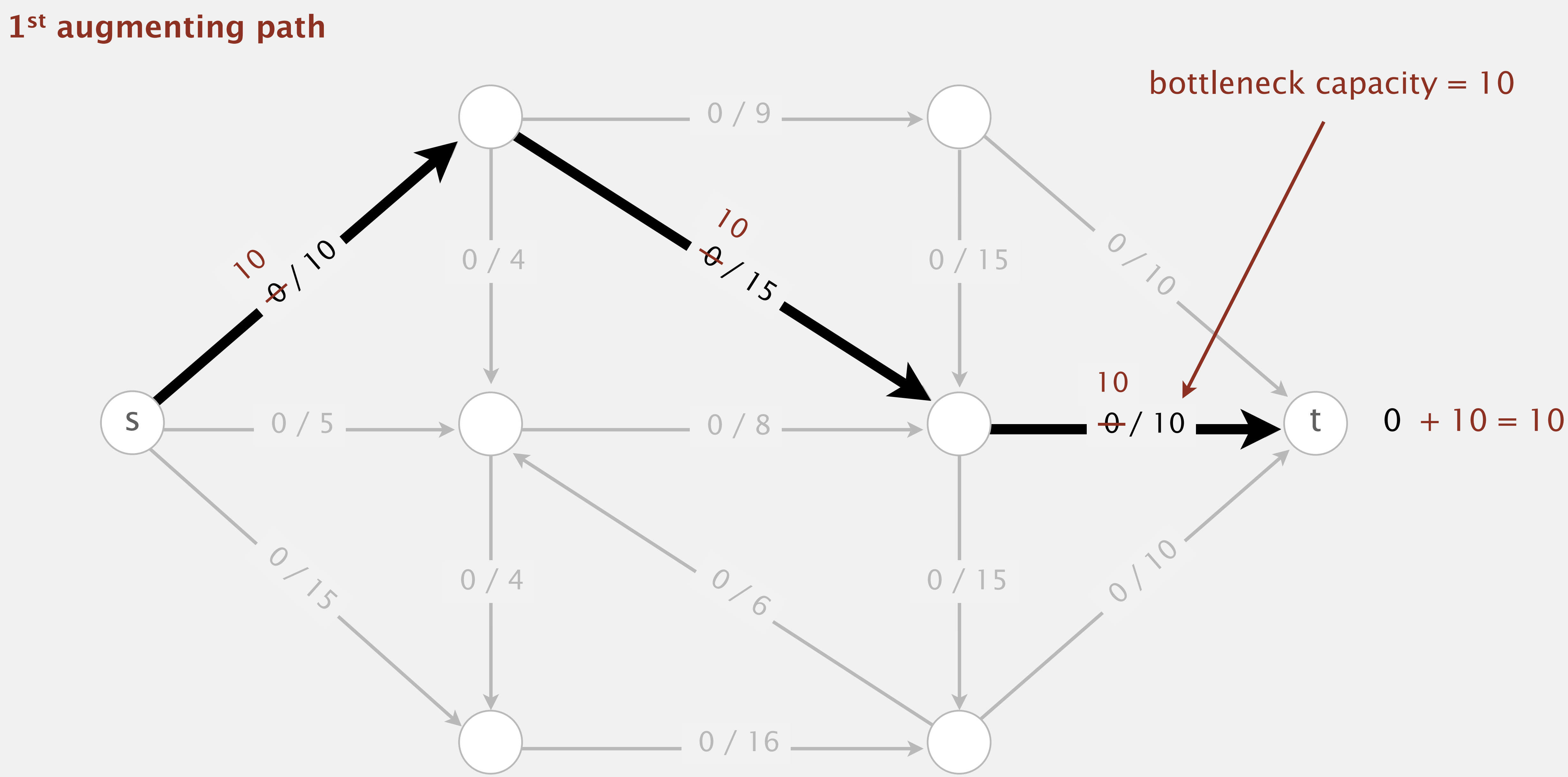
Initialization. Start with 0 flow.



Idea: increase flow along augmenting paths

Augmenting path. Find an undirected path from s to t such that:

- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

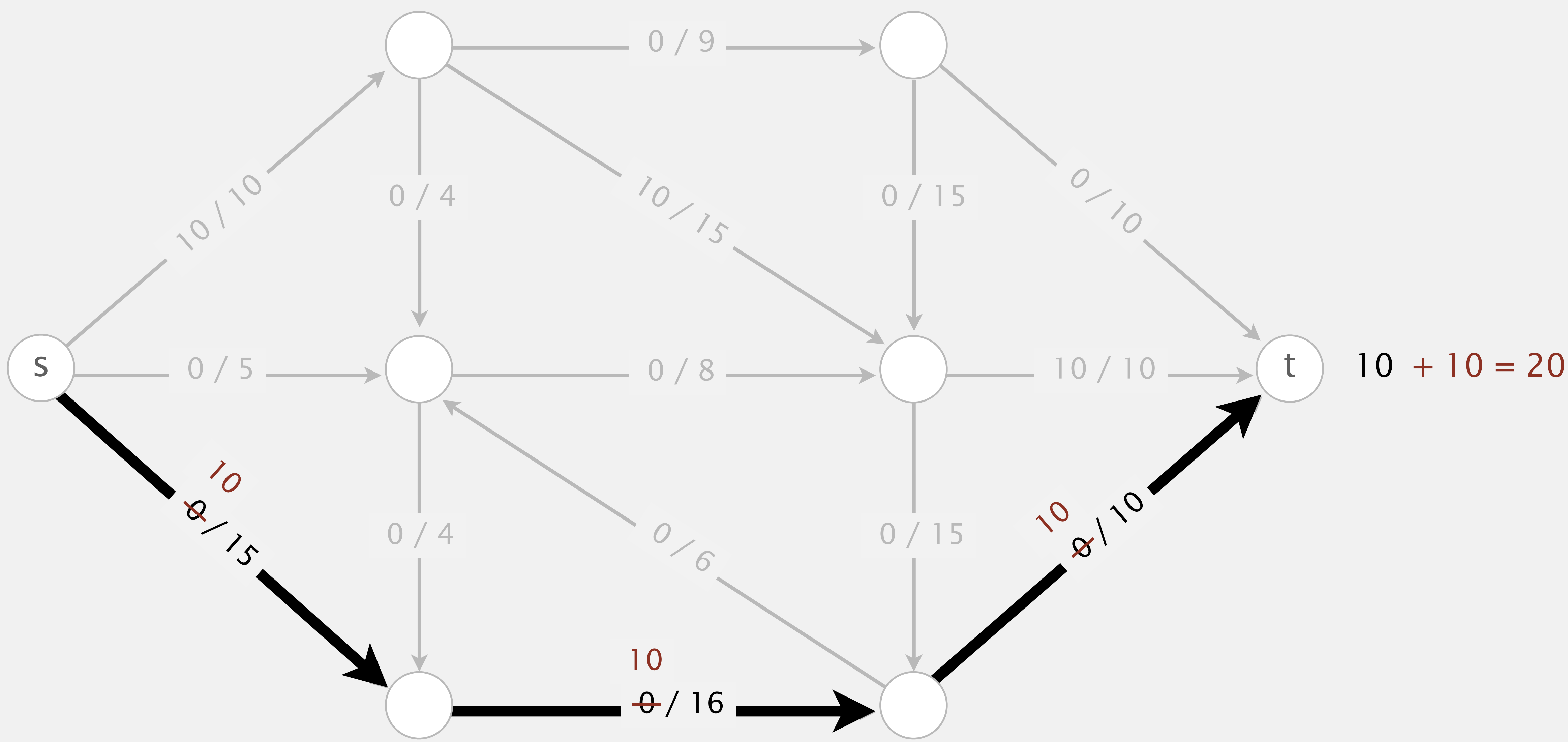


Idea: increase flow along augmenting paths

Augmenting path. Find an undirected path from s to t such that:

- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

2nd augmenting path



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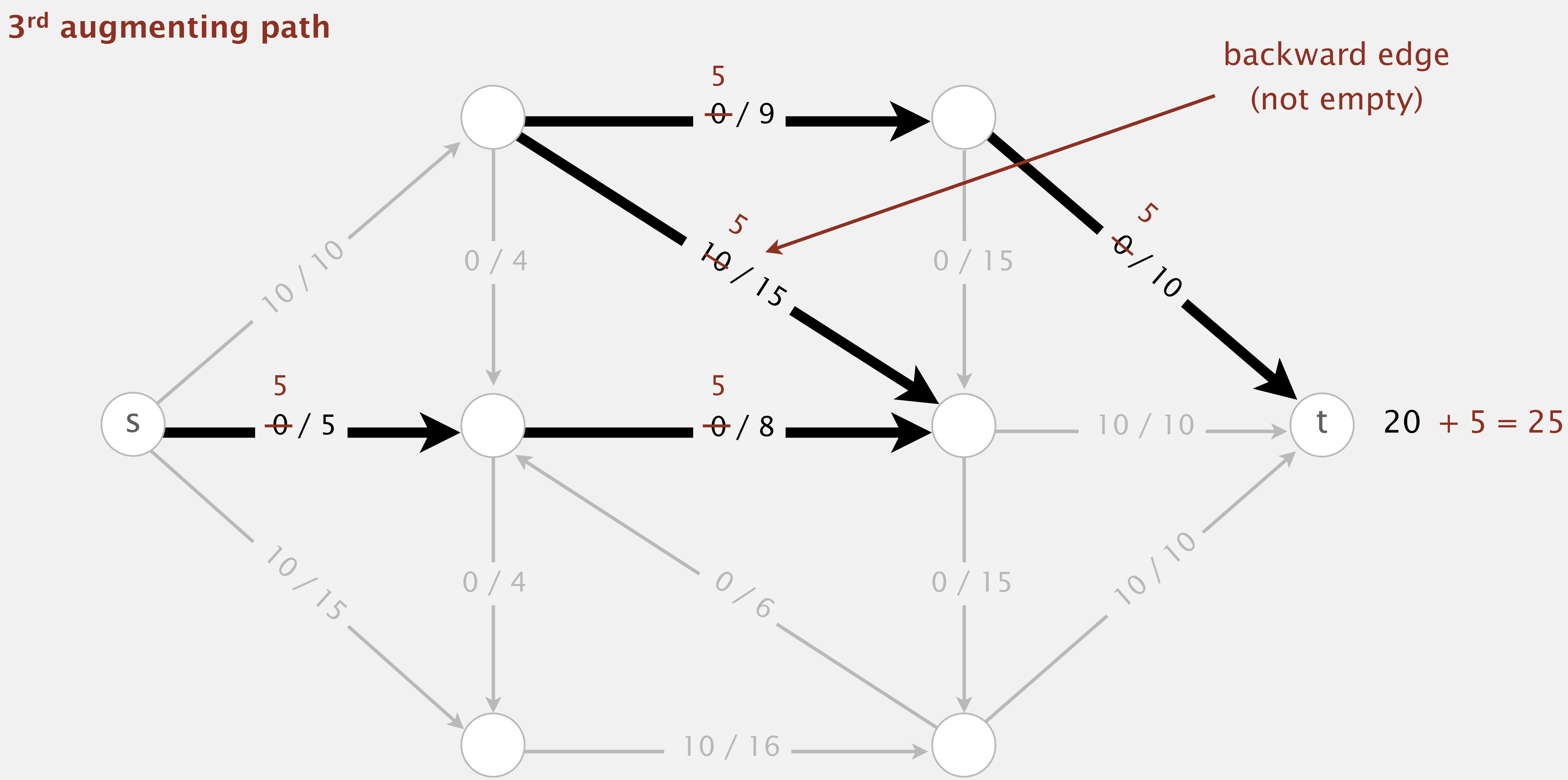
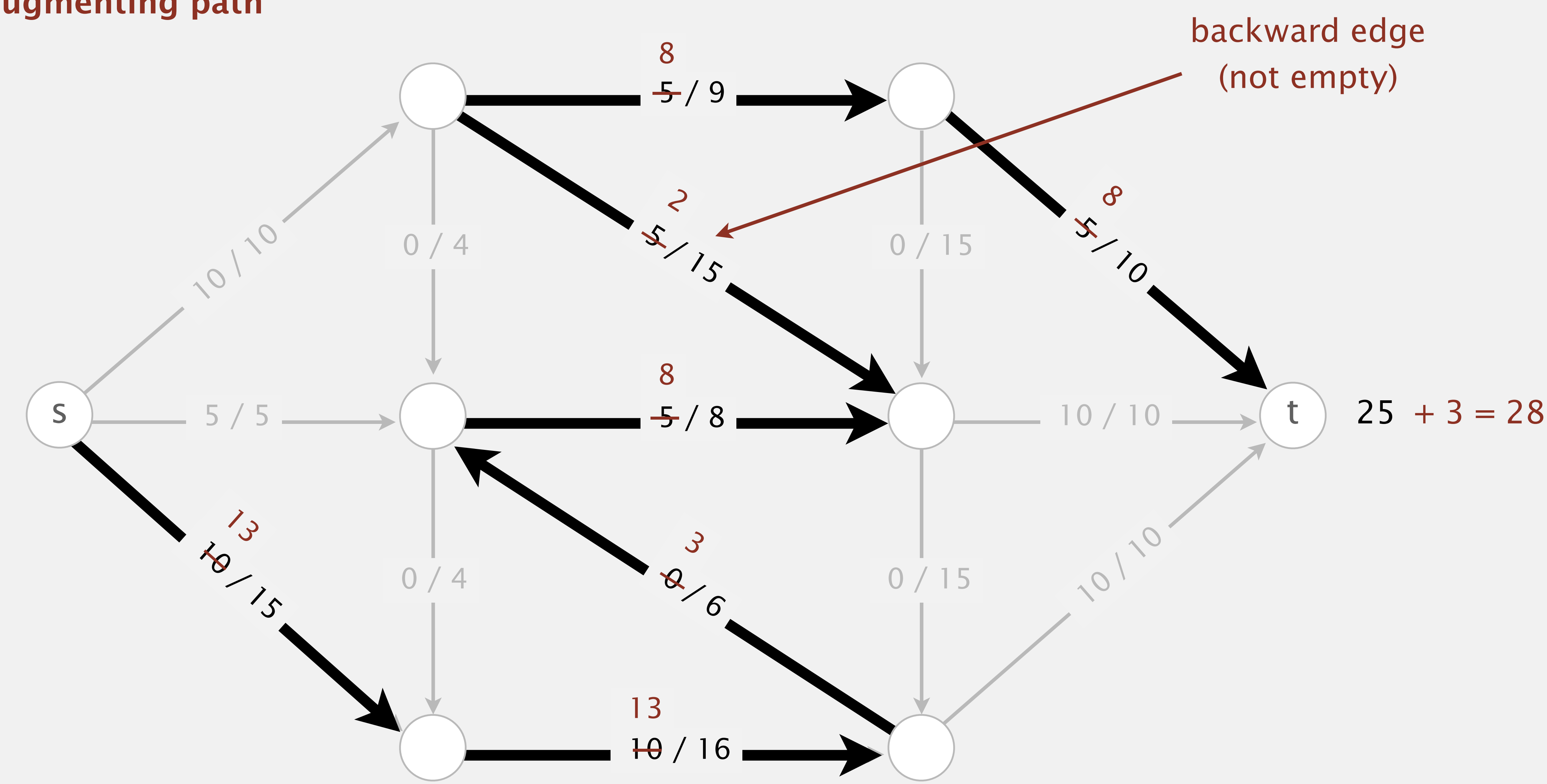


Idea: increase flow along augmenting paths

Augmenting path. Find an undirected path from s to t such that:

- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

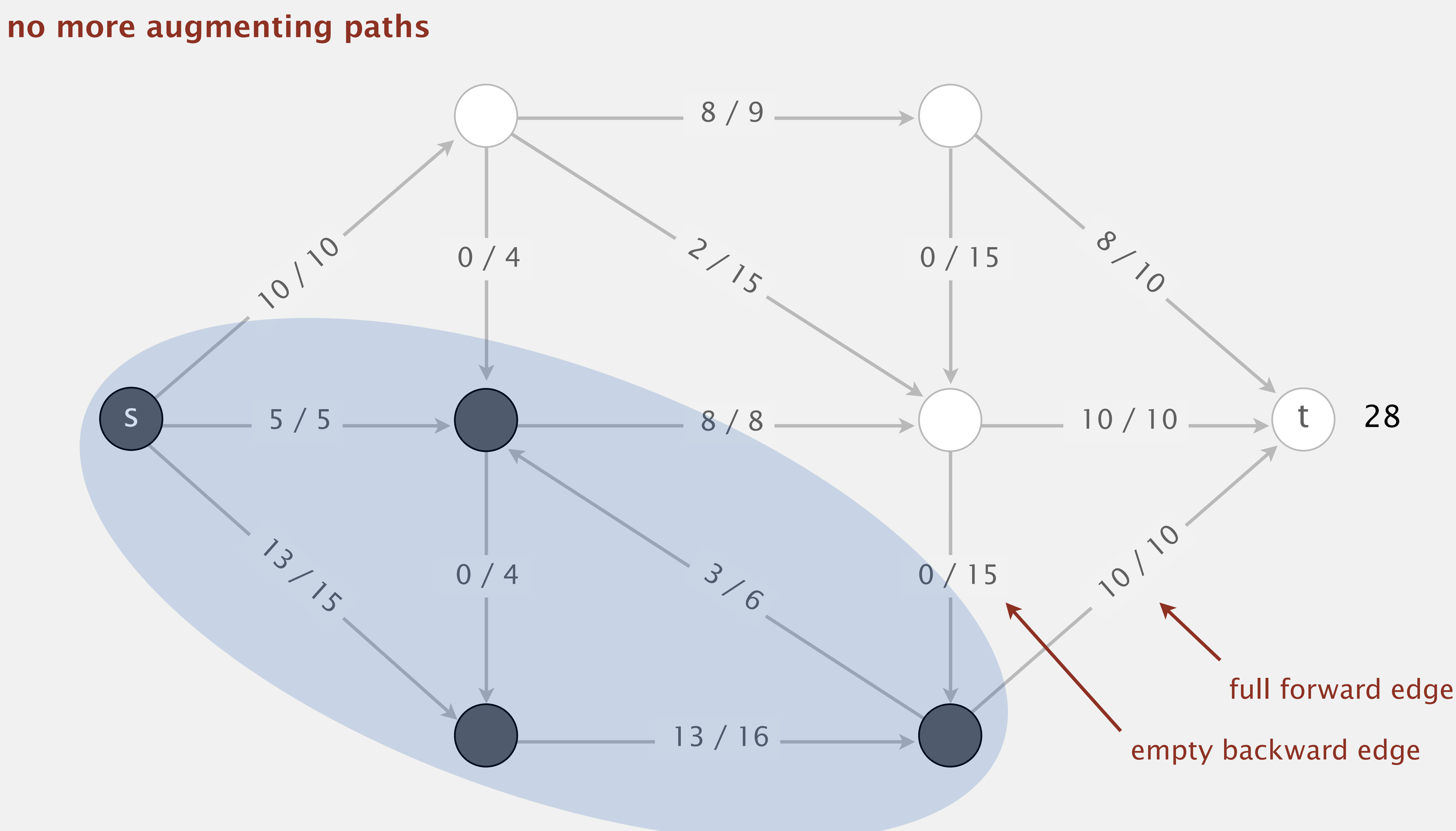
4th augmenting path



Idea: increase flow along augmenting paths

Termination. All paths from s to t are blocked by either a

- Full forward edge.
- Empty backward edge.



Maxflow: quiz 2

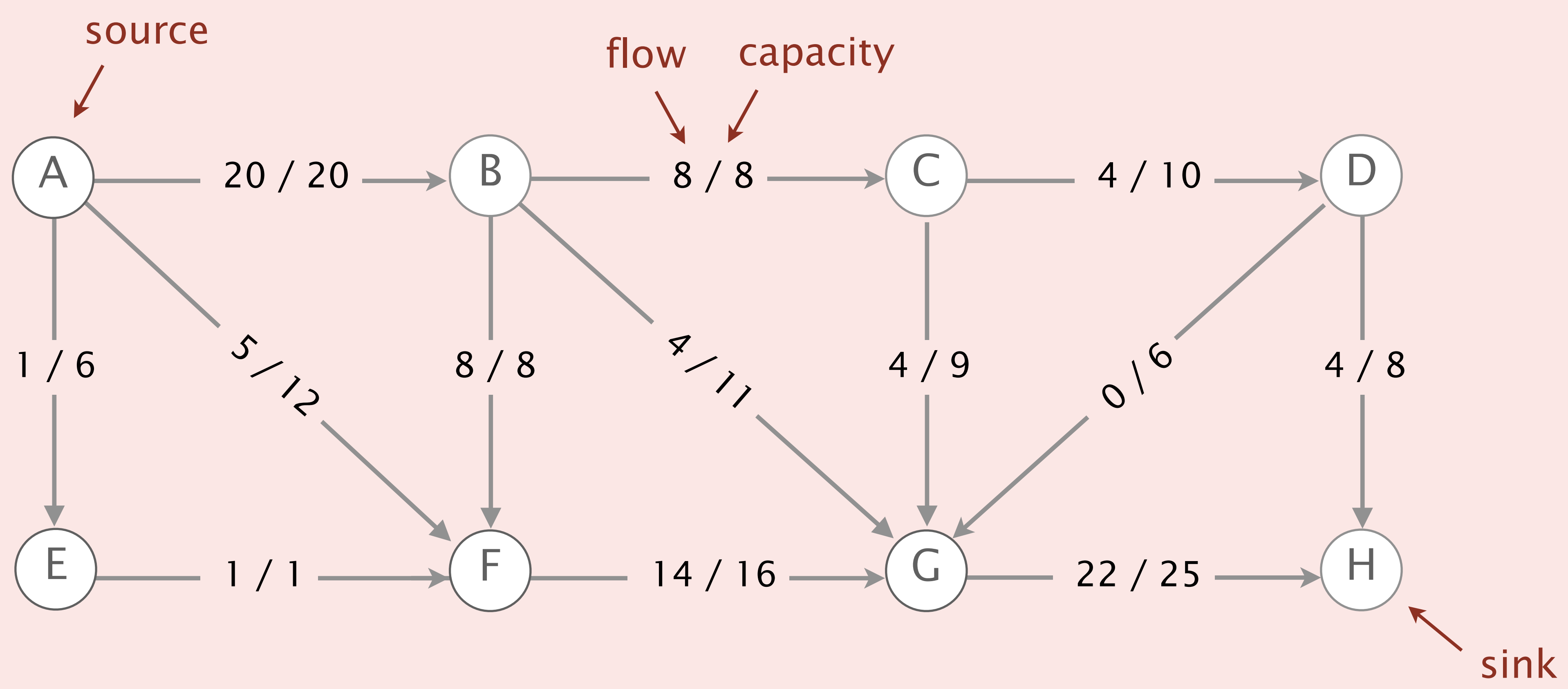
Which is an augmenting path of highest bottleneck capacity?

A. $A \rightarrow F \rightarrow G \rightarrow H$

B. $A \rightarrow F \rightarrow B \rightarrow G \rightarrow H$

C. $A \rightarrow F \rightarrow B \rightarrow G \rightarrow D \rightarrow H$

D. *I don't know.*



Ford-Fulkerson algorithm

Ford–Fulkerson algorithm

Start with 0 flow.

While there exists an augmenting path:

- **find an augmenting path**
- **compute bottleneck capacity**
- **increase flow on that path by bottleneck capacity**

Fundamental questions.

- How to compute a mincut?
- How to find an augmenting path?
- If FF terminates, does it always compute a maxflow?
- Does FF always terminate? If so, after how many augmentations?

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6.4 Maximum Flow

- introduction
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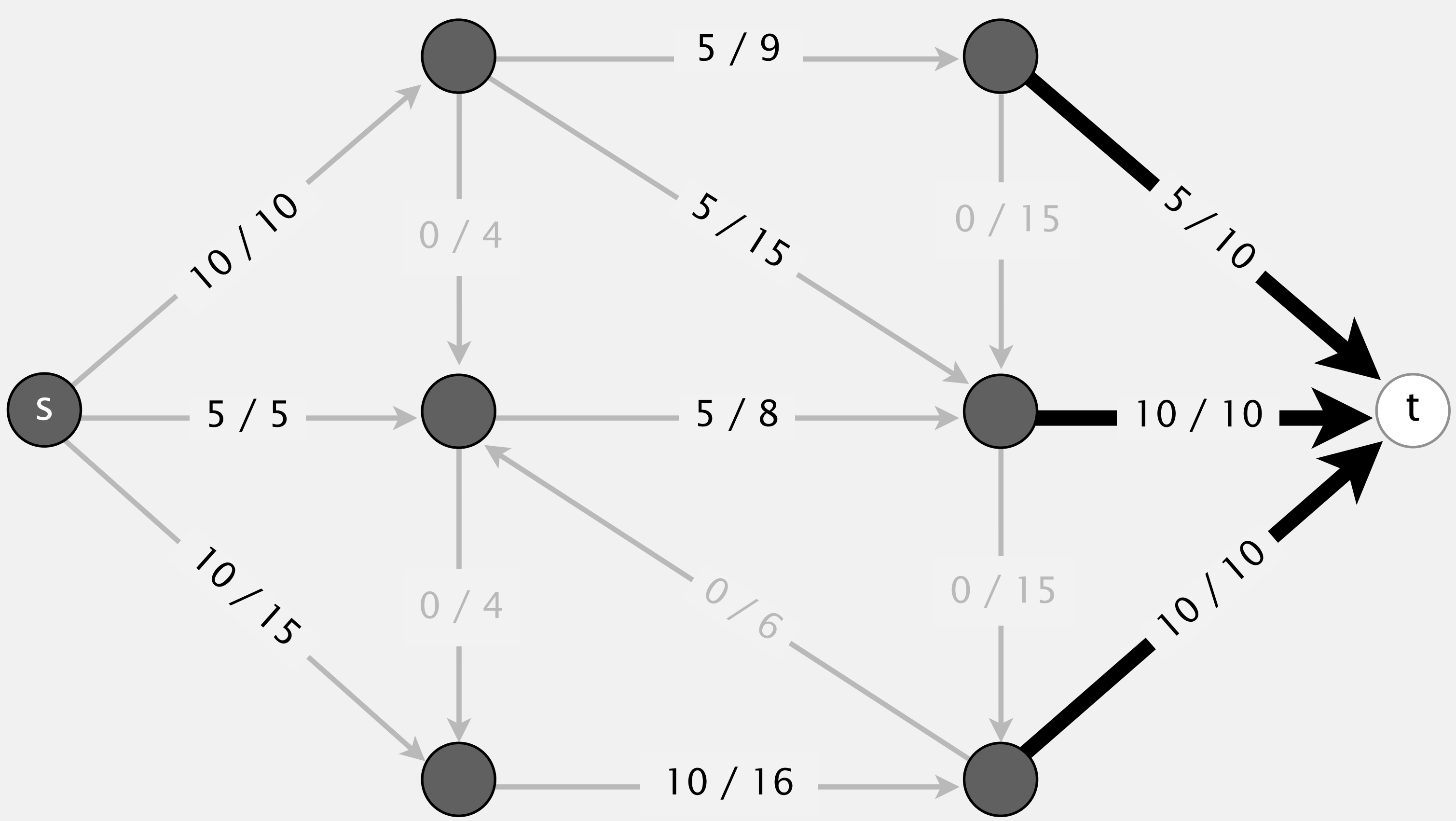
Relationship between flows and cuts

Def. The net flow across a cut (A, B) is the sum of the flows on its edges from A to B minus the sum of the flows on its edges from B to A.

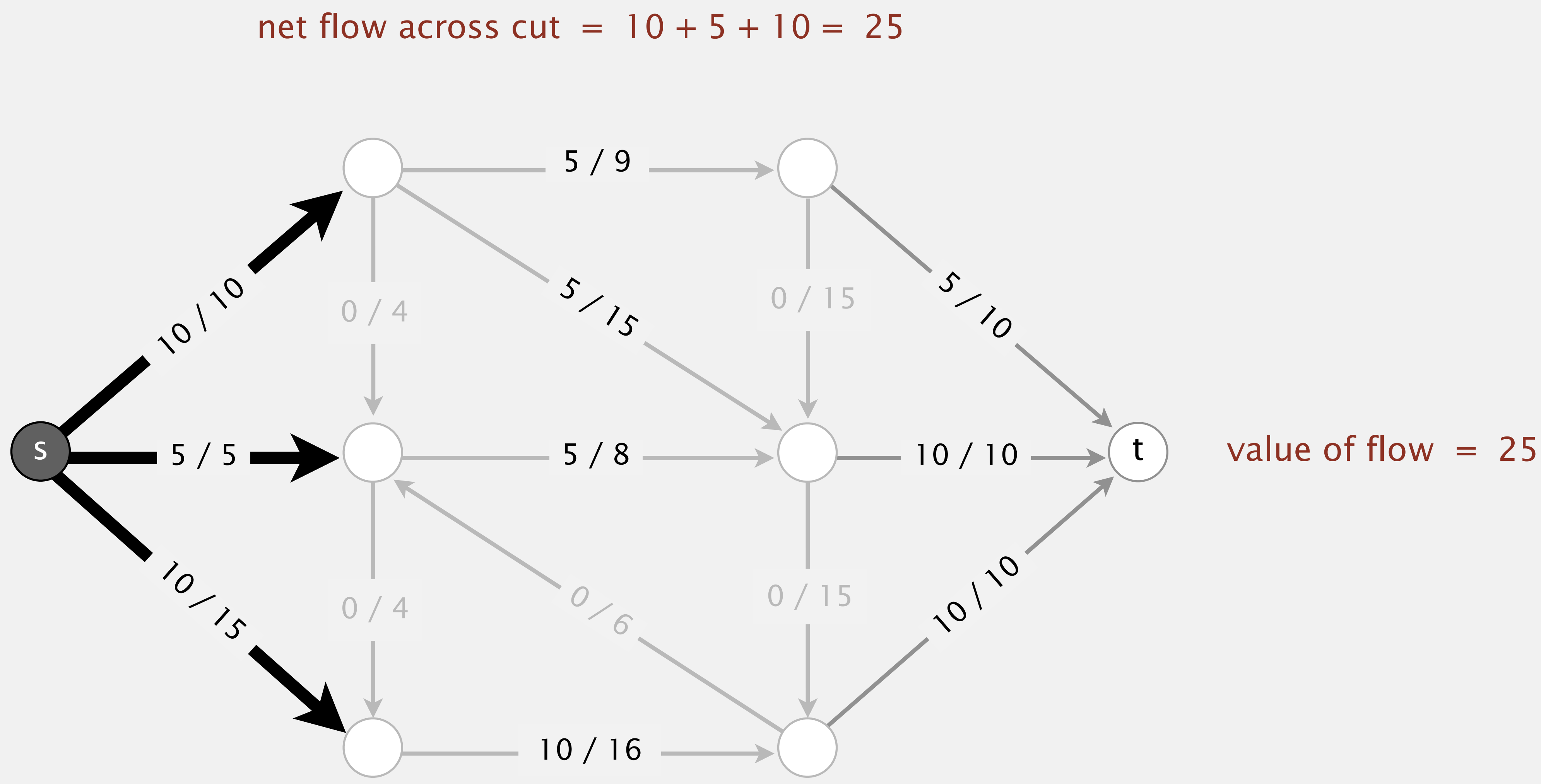
net flow across cut = 5 + 10 + 10 = 25



value of flow = 25

Relationship between flows and cuts

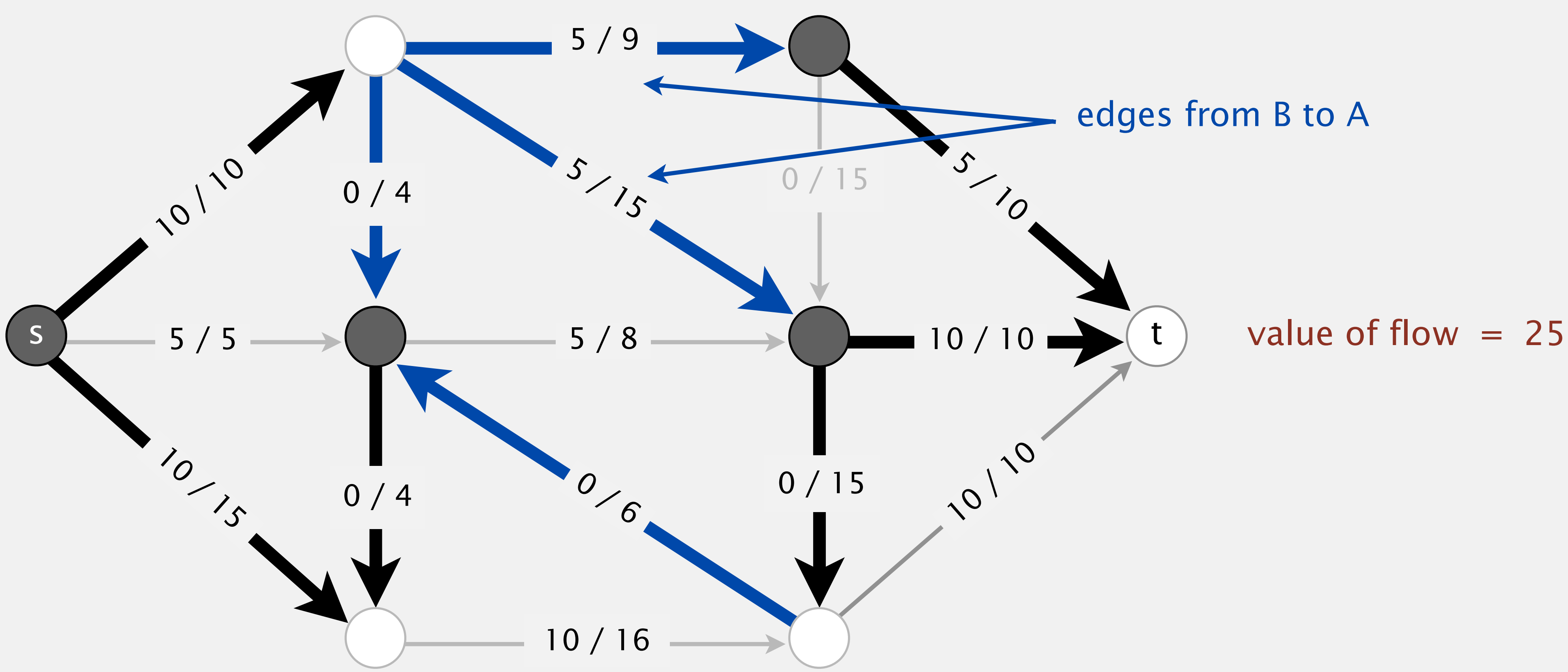
Def. The **net flow across** a cut (A, B) is the sum of the flows on its edges from A to B minus the sum of the flows on its edges from B to A.



Relationship between flows and cuts

Def. The net flow across a cut (A, B) is the sum of the flows on its edges from A to B minus the sum of the flows on its edges from B to A.

net flow across cut = (10 + 10 + 5 + 10 + 0 + 0) − (5 + 5 + 0 + 0) = 25



Relationship between flows and cuts

Flow-value lemma. Let f be any flow and let (A, B) be any cut. Then, the net flow across (A, B) equals the value of f.

Intuition. Conservation of flow.

Pf. By induction on the size of B.

- Base case: $B = \{ t \}$.
- Induction step: remains true by local equilibrium when moving any vertex from A to B.

Corollary. Outflow from s = inflow to t = value of flow.

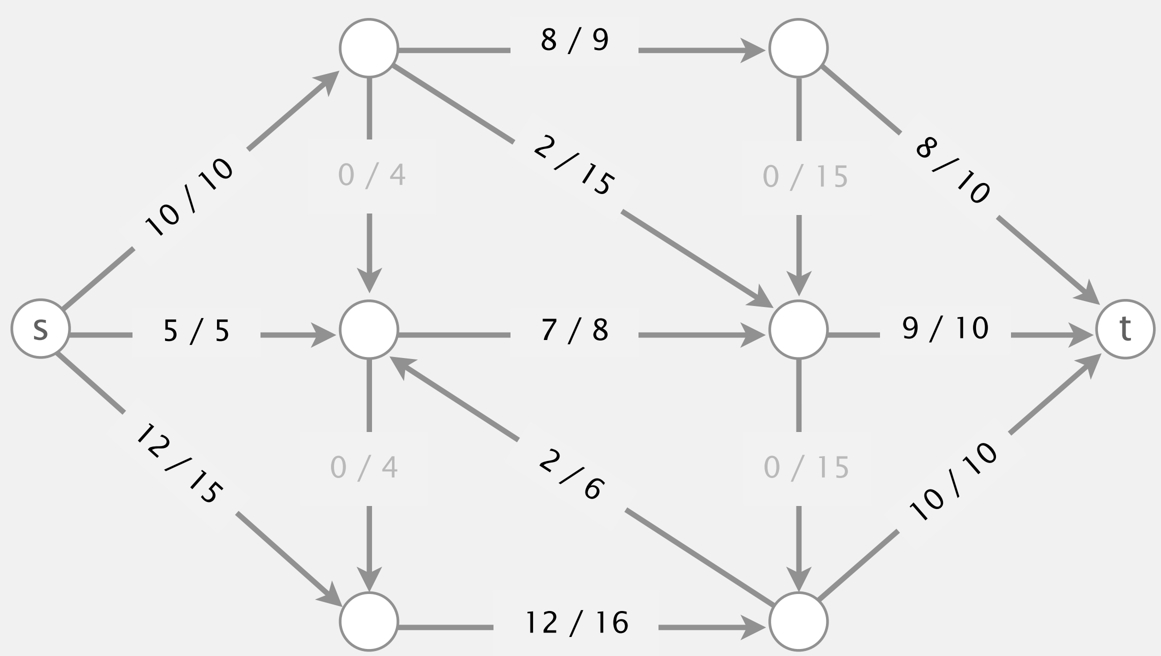
Relationship between flows and cuts

Weak duality. Let f be any flow and let (A, B) be any cut.
Then, the value of the flow $\leq$ the capacity of the cut.

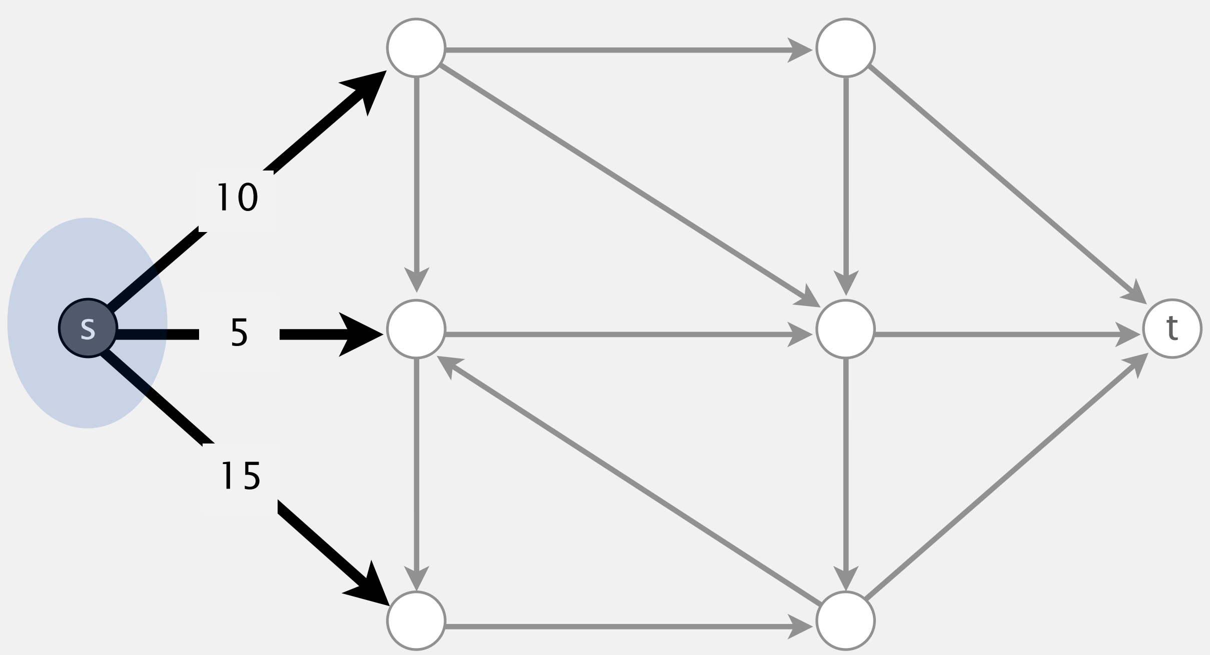
Pf. Value of flow f = net flow across cut (A, B) $\leq$ capacity of cut (A, B).

flow-value lemma

flow bounded by capacity



value of flow = 27



capacity of cut = 30

Maxflow-mincut theorem

Augmenting path theorem. A flow f is a maxflow iff no augmenting paths.

Maxflow-mincut theorem. Value of the maxflow = capacity of mincut.

Pf. The following three conditions are equivalent for any flow f:

i. There exists a cut whose capacity equals the value of the flow f.

ii. f is a maxflow.

iii. There is no augmenting path with respect to f.

[i ⇒ ii]

- Suppose that (A, B) is a cut with capacity equal to the value of f.
- Then, the value of any flow $f' \underbrace{\leq}_{\text{weak duality}}$ capacity of $(A, B) \underbrace{=}_{\text{by assumption}}$ value of f.
- Thus, f is a maxflow.

Maxflow-mincut theorem

Augmenting path theorem. A flow f is a maxflow iff no augmenting paths.

Maxflow-mincut theorem. Value of the maxflow = capacity of mincut.

Pf. The following three conditions are equivalent for any flow f:

i. There exists a cut whose capacity equals the value of the flow f.

ii. f is a maxflow.

iii. There is no augmenting path with respect to f.

[ii ⇒ iii] We prove contrapositive: ~iii ⇒ ~ii.

- Suppose that there is an augmenting path with respect to f.
- Can improve flow f by sending flow along this path.
- Thus, f is not a maxflow.

Maxflow-mincut theorem

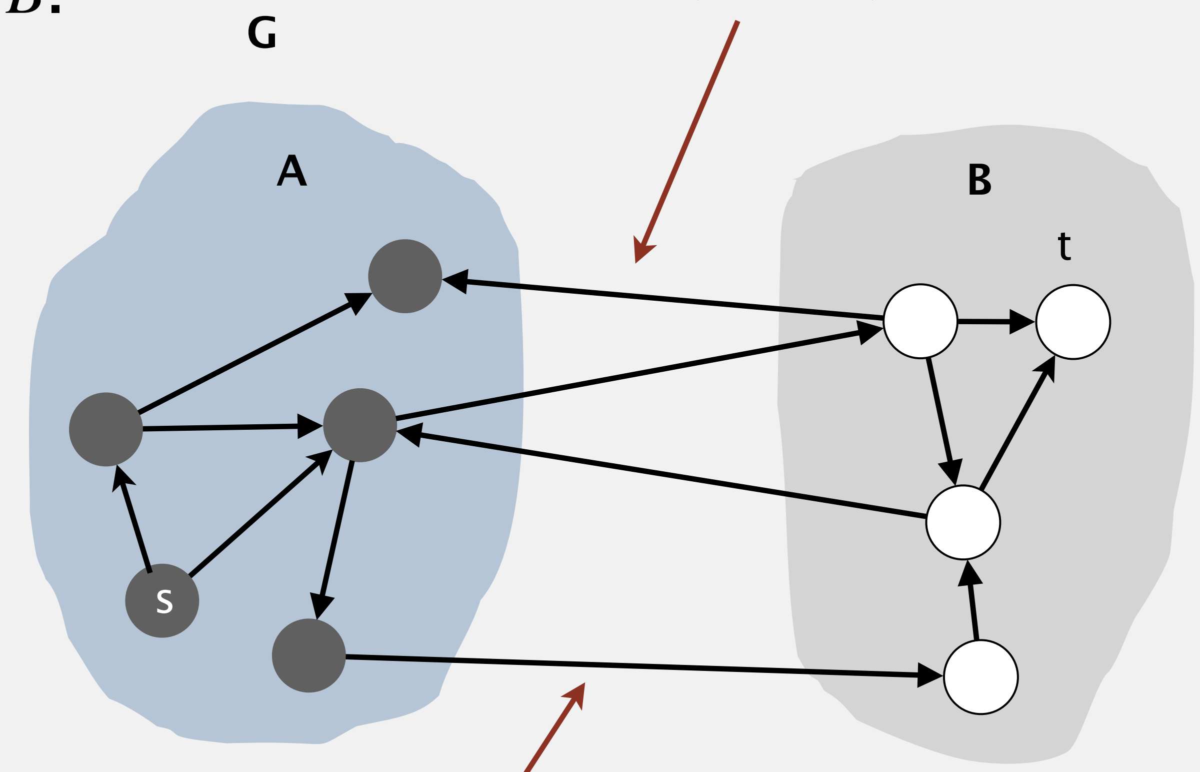
[iii ⇒ i]

- Let f be a flow with no augmenting paths.
- Let A be set of vertices connected to s by an undirected path with no full forward or empty backward edges
- By definition of cut A, s is in A.
- By definition of cut A and flow f, t is in B.
- Capacity of cut $=$ net flow across cut

  $=$ value of flow f.

flow-value lemma

backward edge from B to A (flow = 0)

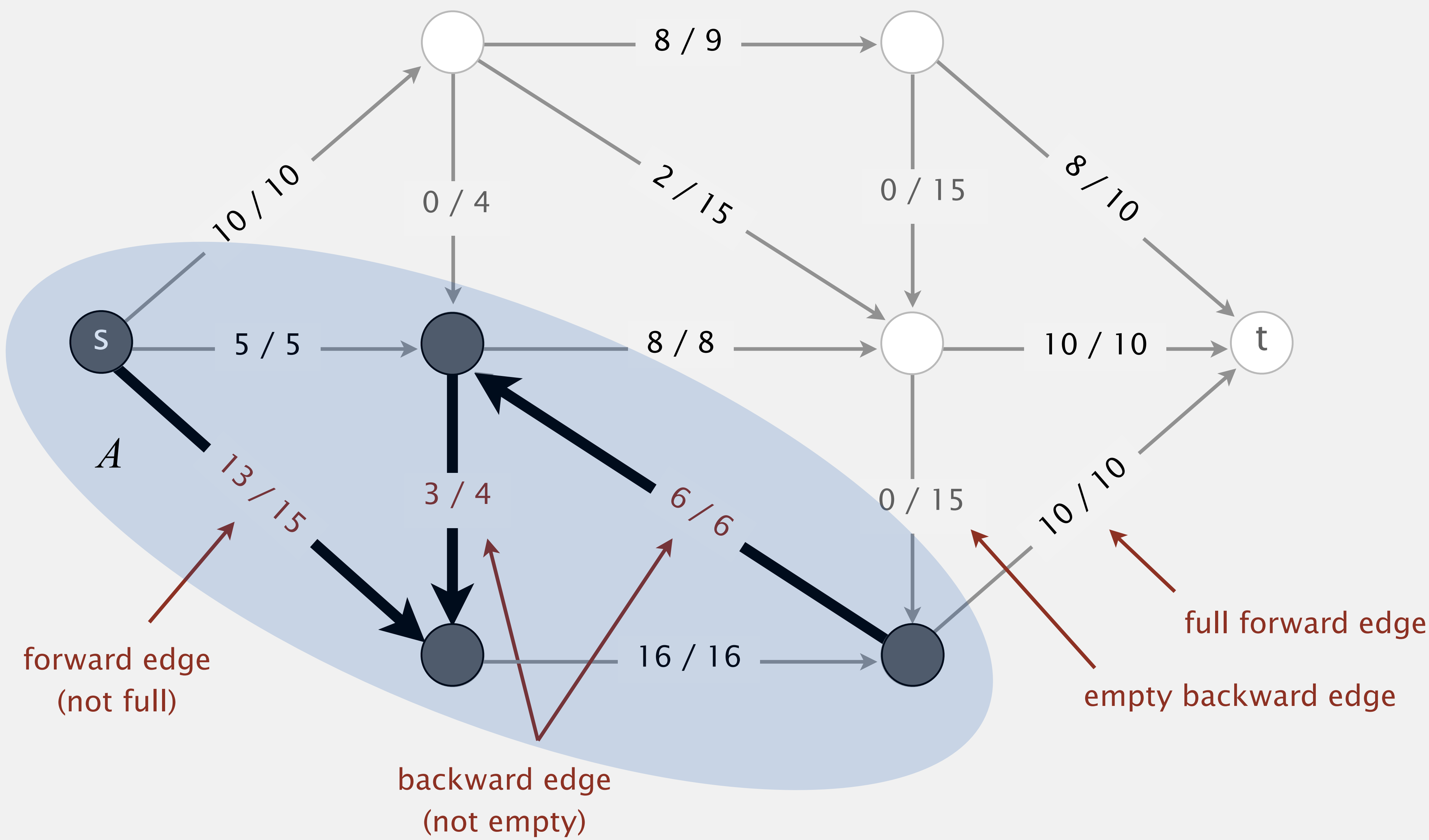


forward edge from A to B (flow = capacity)

Computing a mincut from a maxflow

To compute mincut (A, B) from maxflow f :

- By augmenting path theorem, no augmenting paths with respect to f.
- Compute A = set of vertices connected to s by an undirected path with no full forward or empty backward edges.



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6.4 Maximum Flow

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Ford-Fulkerson algorithm

Ford–Fulkerson algorithm

Start with 0 flow.

While there exists an augmenting path:

- **find an augmenting path**
- **compute bottleneck capacity**
- **increase flow on that path by bottleneck capacity**

Fundamental questions.

- How to compute a mincut? Easy. ✔
- How to find an augmenting path? BFS works well.
- If FF terminates, does it always compute a maxflow? Yes. ✔
- Does FF always terminate? If so, after how many augmentations?

yes, provided edge capacities are integers
(or augmenting paths are chosen carefully)

requires clever analysis

Ford-Fulkerson algorithm with integer capacities

Important special case. Edge capacities are integers between 1 and U.

flow on each edge is an integer

Invariant. The flow is integral throughout Ford-Fulkerson.

Pf. [by induction]

- Bottleneck capacity is an integer.
- Flow on an edge increases/decreases by bottleneck capacity.

Proposition. Number of augmentations ≤ the value of the maxflow.

Pf. Each augmentation increases the value by at least 1.

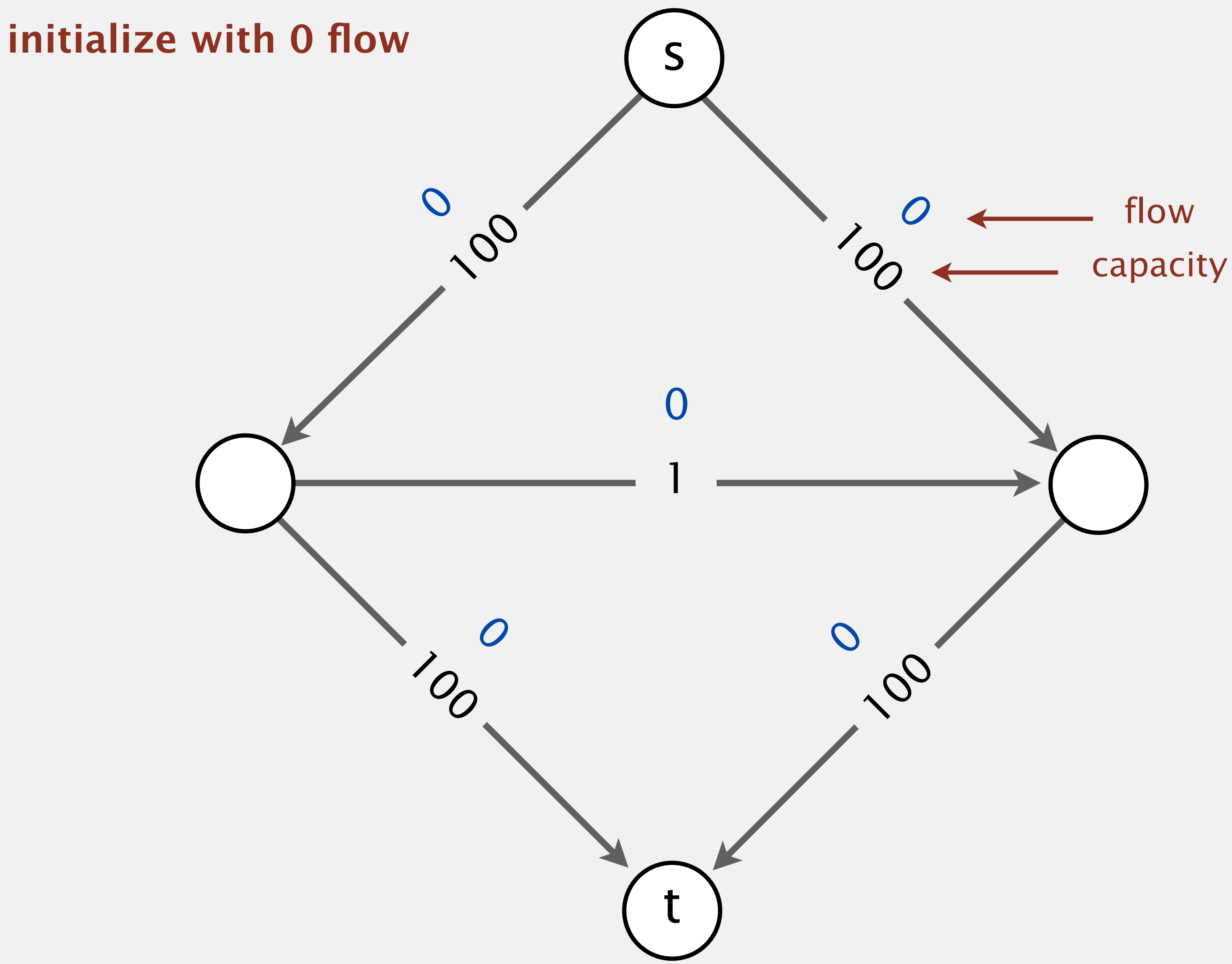
critical for some applications (stay tuned)

Integrality theorem. There exists an integral maxflow.

Pf. Ford-Fulkerson terminates and maxflow that it finds is integer-valued.

Bad case for Ford-Fulkerson

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.



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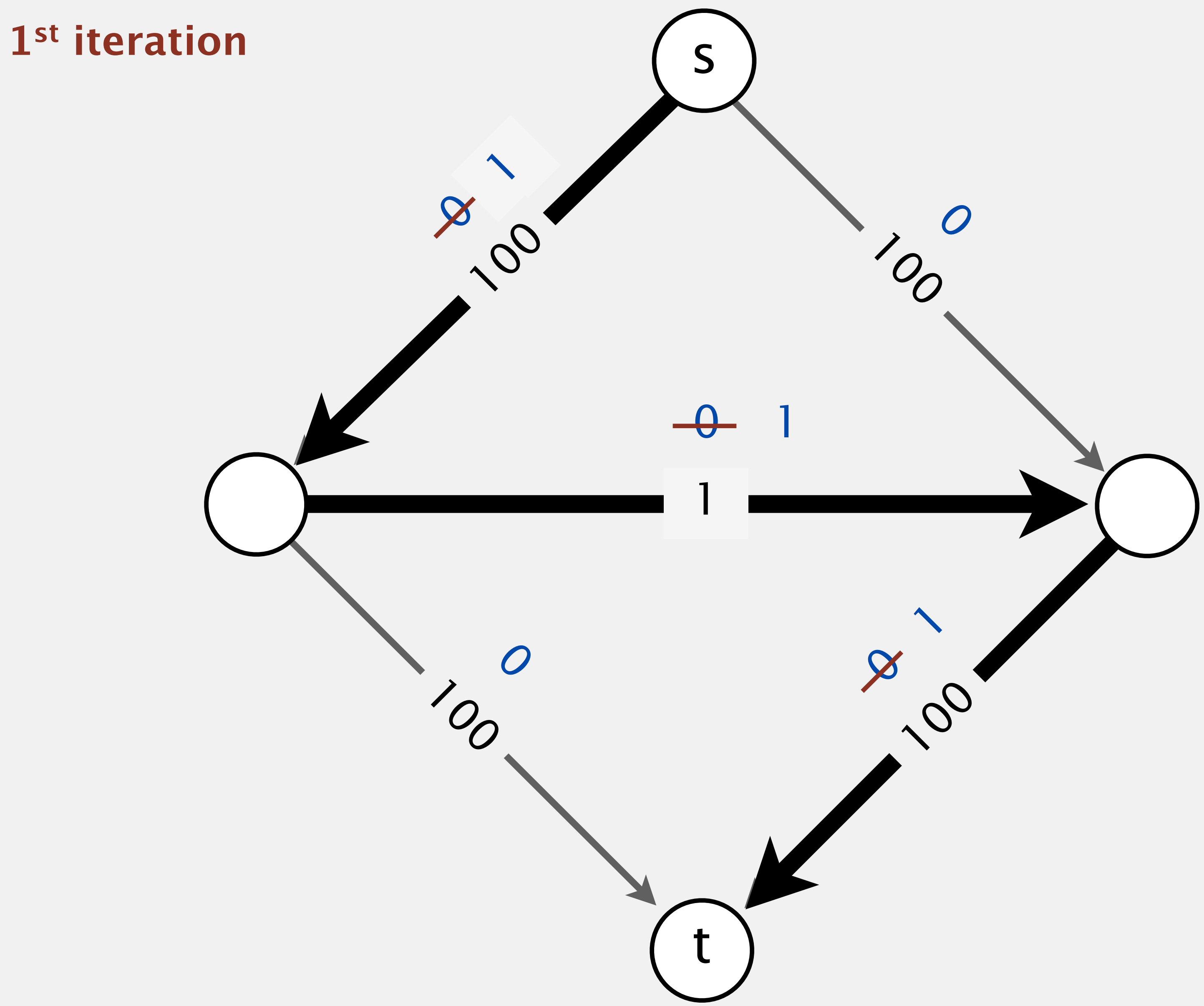

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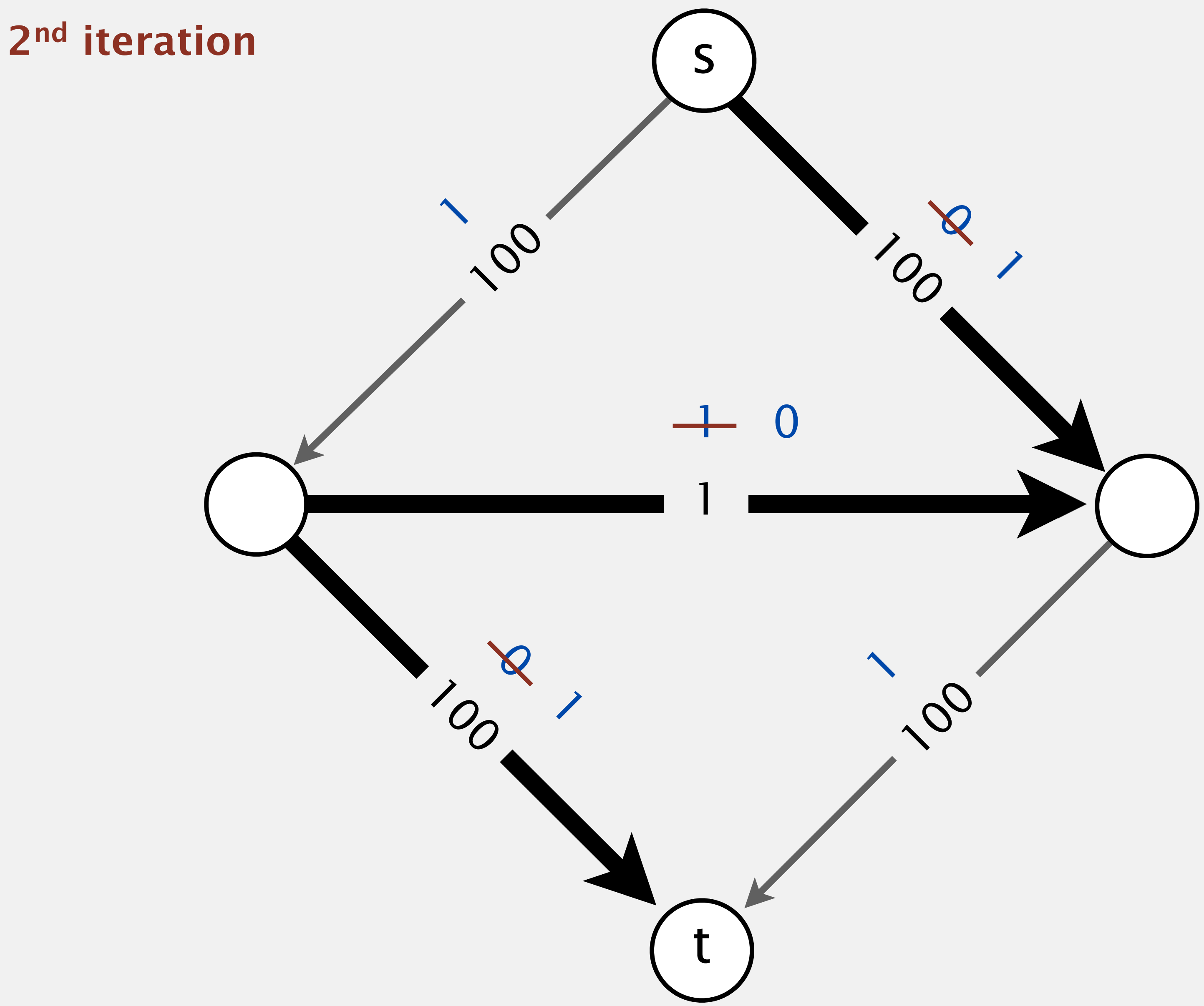

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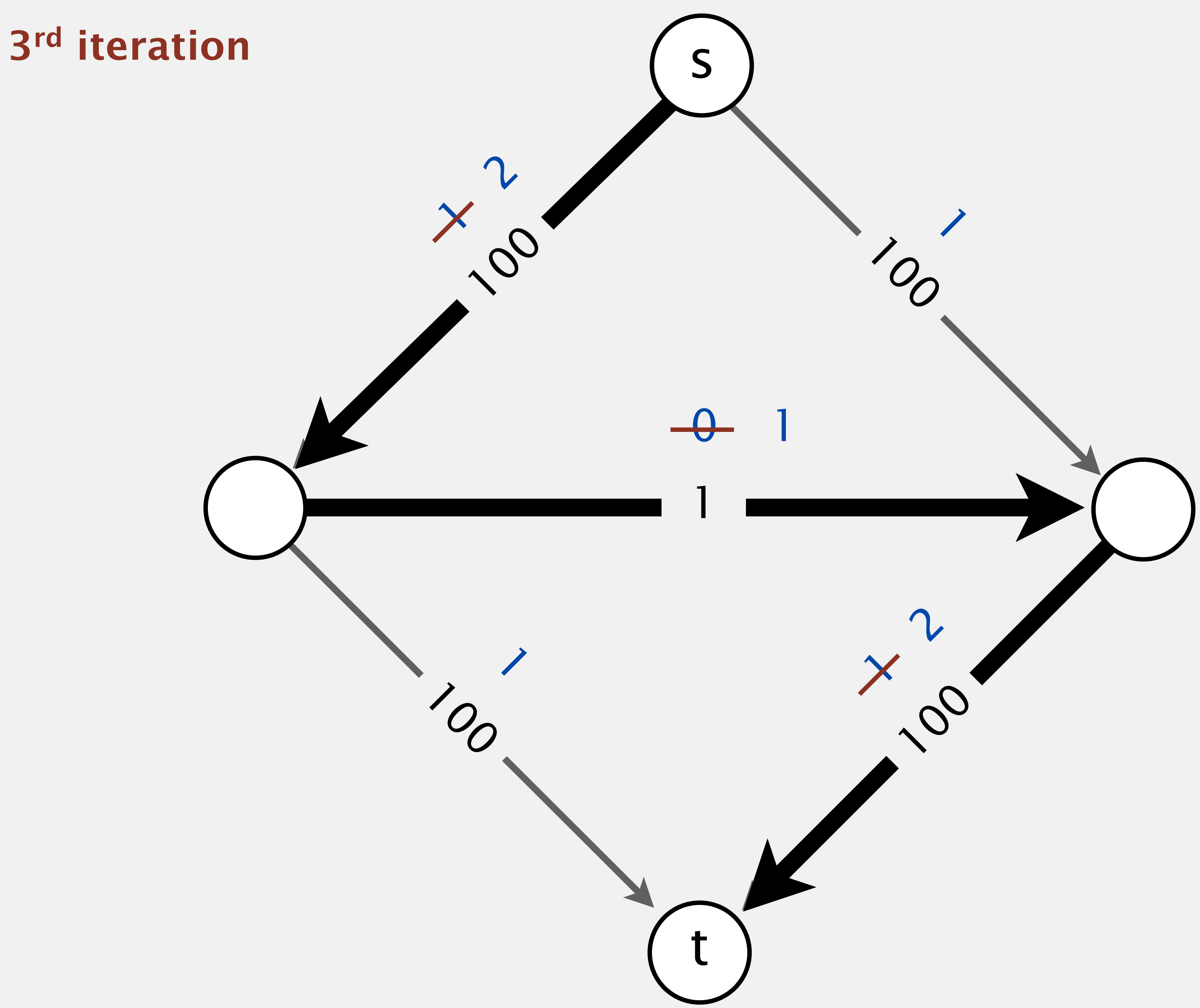

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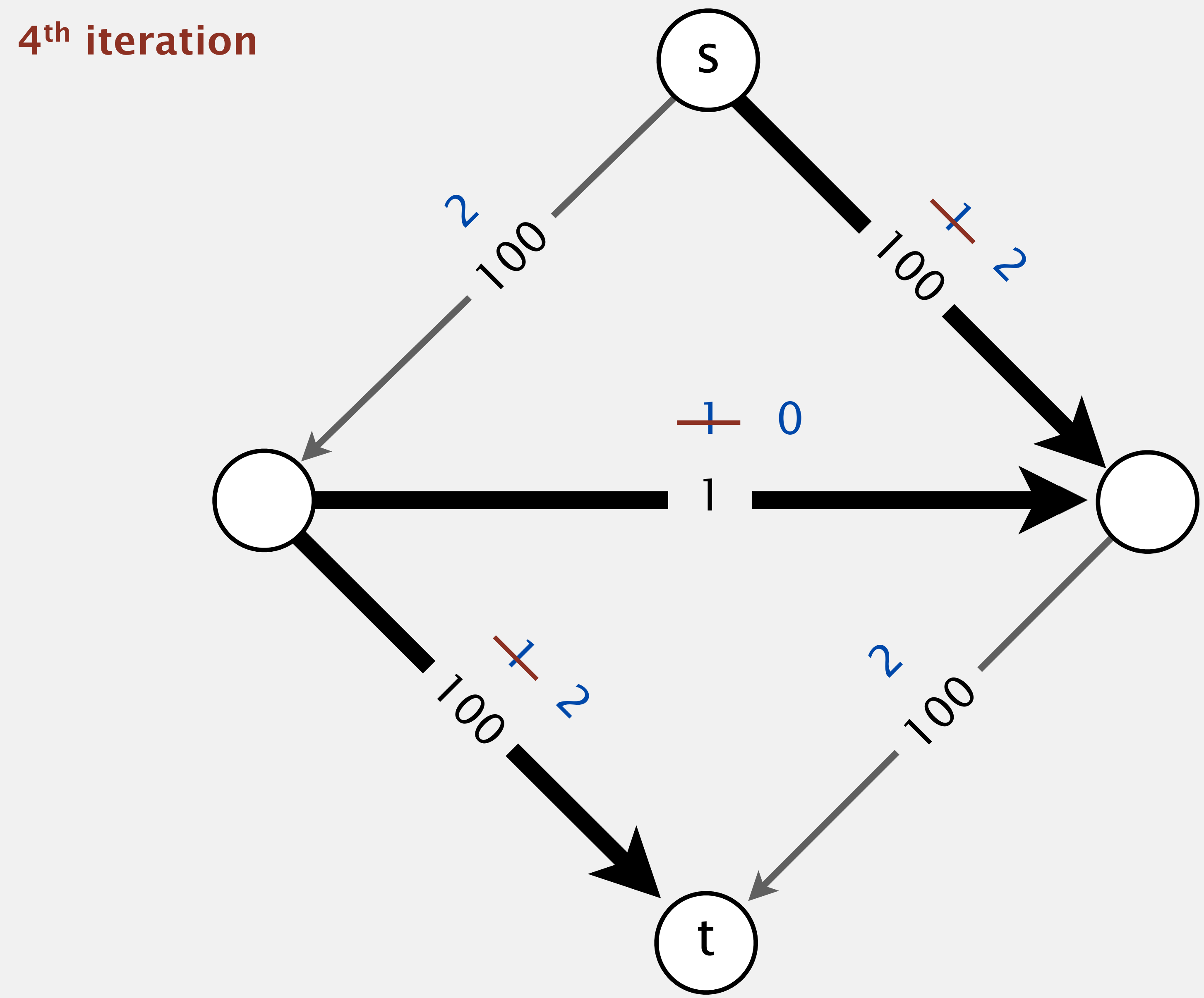

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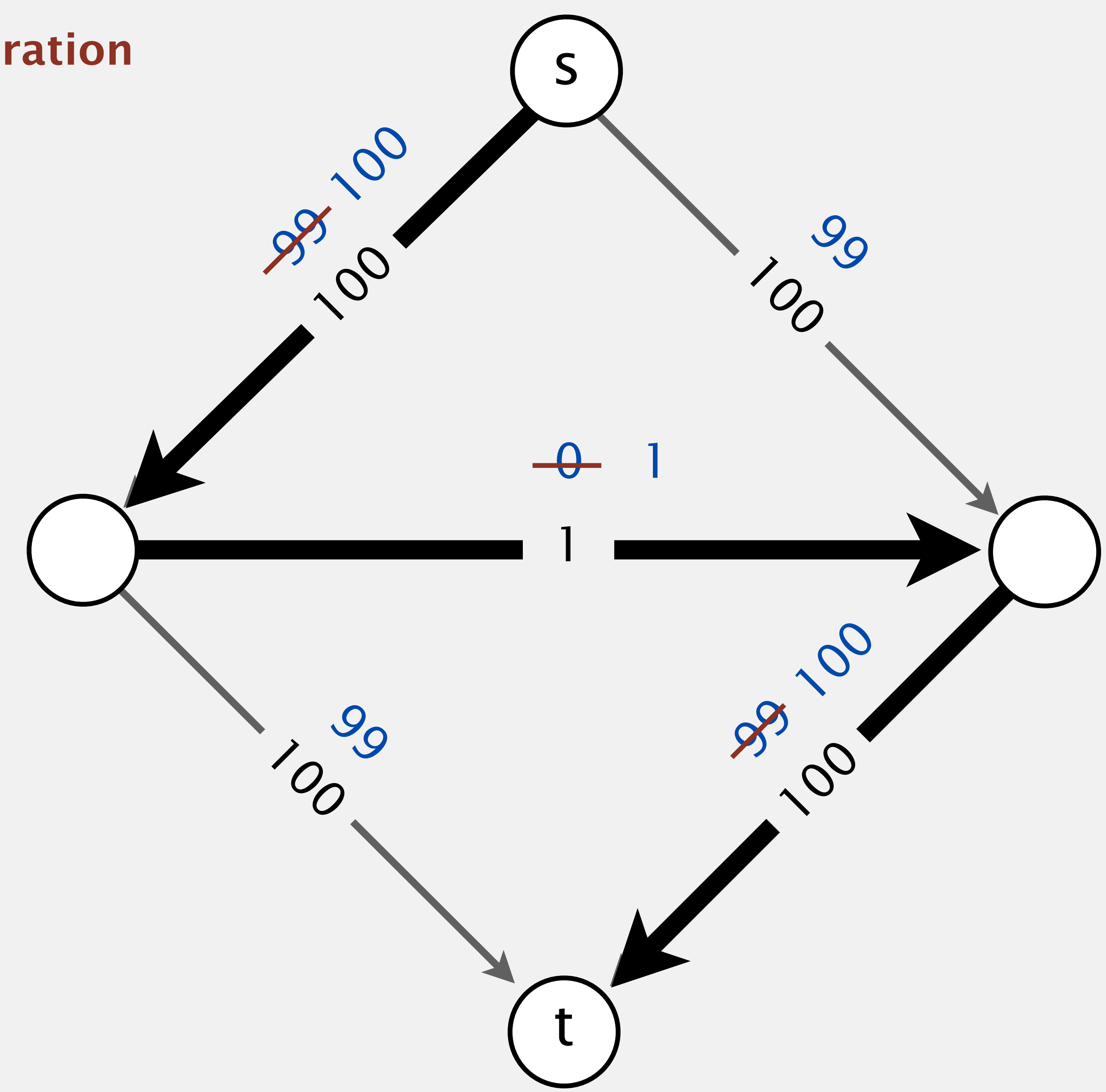
Bad case for Ford-Fulkerson

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.

199th iteration



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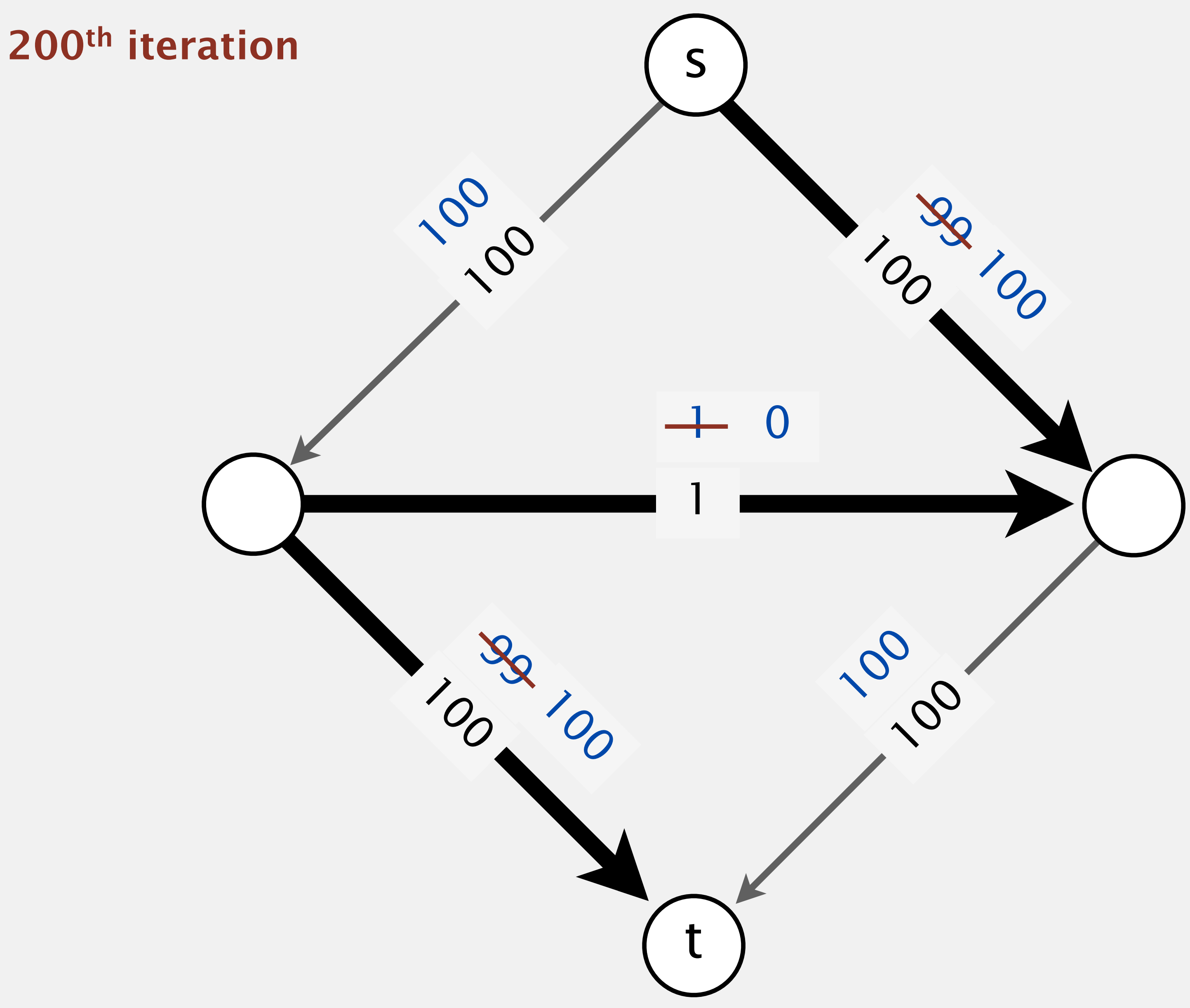
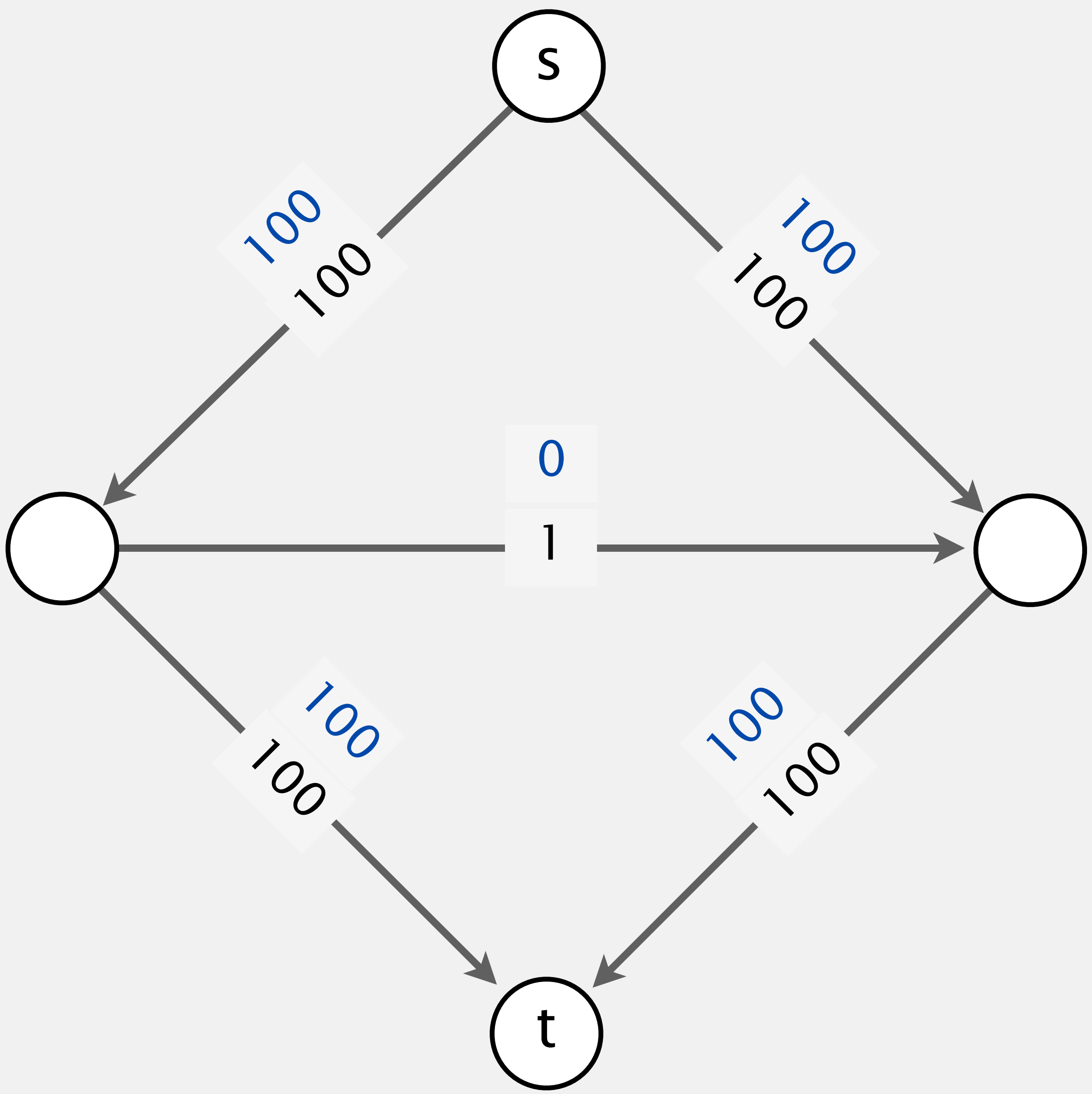


Bad case for Ford-Fulkerson

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.

can be exponential in input size

Good news. This case is easily avoided. [use shortest/fattest path]



How to choose augmenting paths?

Choose augmenting paths with:

- Shortest path: fewest number of edges.
- Fattest path: max bottleneck capacity.

Theoretical Improvements in Algorithmic Efficiency for Network Flow Problems

JACK EDMONDS

University of Waterloo, Waterloo, Ontario, Canada

AND

RICHARD M. KARP

University of California, Berkeley, California

ABSTRACT. This paper presents new algorithms for the maximum flow problem, the Hitchcock transportation problem, and the general minimum-cost flow problem. Upper bounds on the numbers of steps in these algorithms are derived, and are shown to compare favorably with upper bounds on the numbers of steps required by earlier algorithms.

Edmonds–Karp 1972 (USA)

Dokl. Akad. Nauk SSSR
Tom 194 (1970), No. 4

Soviet Math. Dokl.
Vol. 11 (1970), No. 5

ALGORITHM FOR SOLUTION OF A PROBLEM OF MAXIMUM FLOW IN A NETWORK WITH POWER ESTIMATION

UDC 518.5

E. A. DINIC

Different variants of the formulation of the problem of maximal stationary flow in a network and its many applications are given in [1]. There also is given an algorithm solving the problem in the case where the initial data are integers (or, what is equivalent, commensurable). In the general case this algorithm requires preliminary rounding off of the initial data, i.e. only an approximate solution of the problem is possible. In this connection the rapidity of convergence of the algorithm is inversely proportional to the relative precision.

Dinic 1970 (Soviet Union)

How to choose augmenting paths?

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.

augmenting path	number of paths	implementation
random path	$\leq E\,U$	randomized queue
DFS path	$\leq E\,U$	stack (DFS)
shortest path	$\leq \frac{1}{2}\,E\,V$	queue (BFS)
fattest path	$\leq E \ln(E\,U)$	priority queue

flow network with V vertices, E edges, and integer capacities between 1 and U

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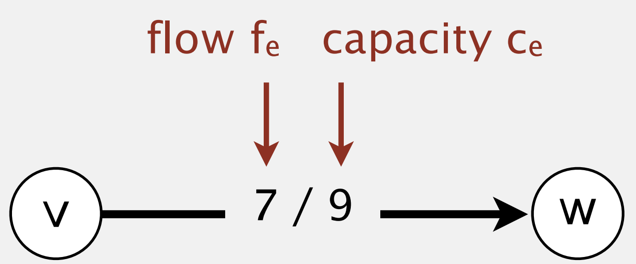
Flow network representation

Flow edge data type. Associate flow f_e and capacity c_e with edge $e = v \rightarrow w$.



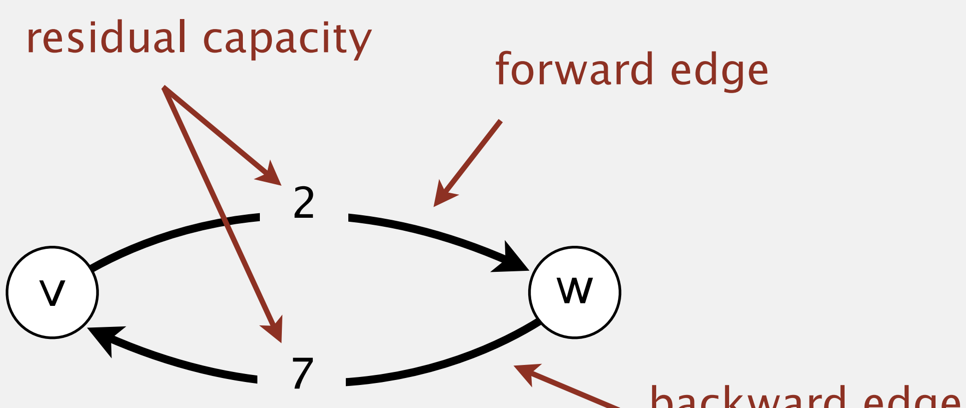
Flow network data type. Must be able to process edge $e = v \rightarrow w$ in either direction: include e in adjacency lists of both v and w.

Residual (spare) capacity.

- Forward edge: residual capacity $= c_e - f_e$.
- Backward edge: residual capacity $= f_e$.

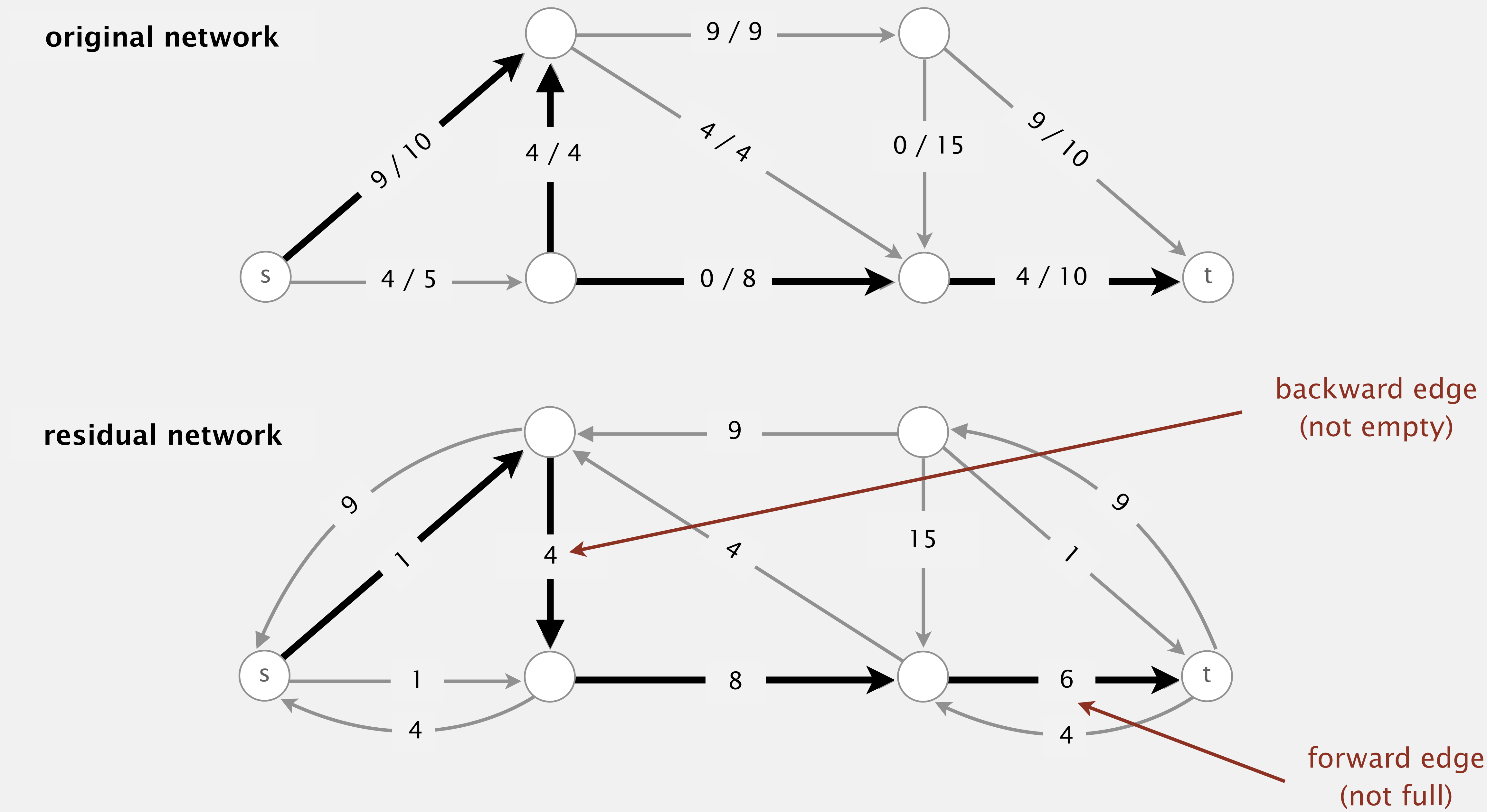
Augment flow.

- Forward edge: add Δ.
- Backward edge: subtract Δ.



Flow network representation

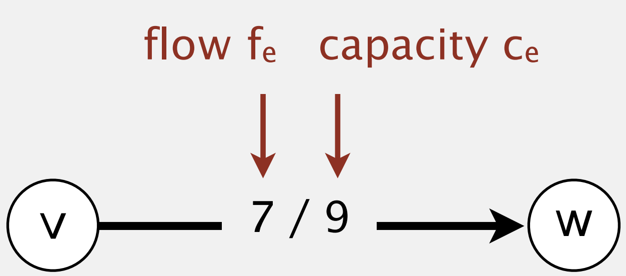
Residual network. A useful view of a flow network. ← includes all edges with positive residual capacity

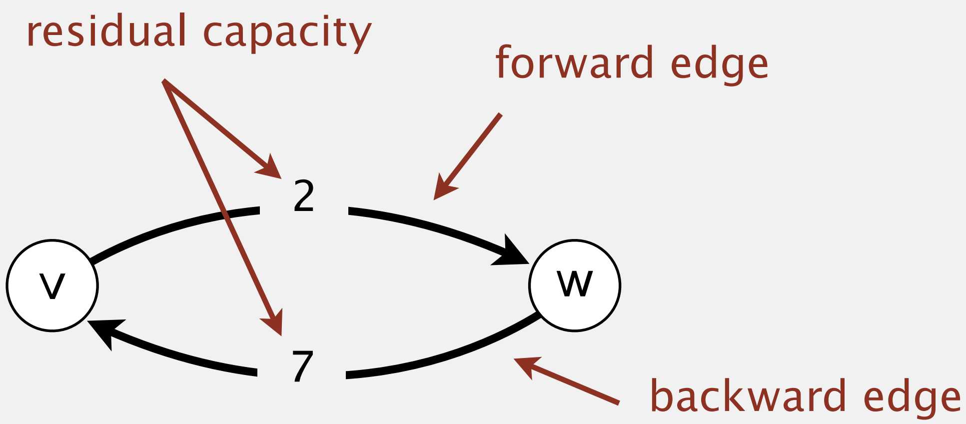


Key point. Augmenting paths in original network are in one-to-one correspondence with directed paths in residual network.

Flow edge API

public class FlowEdge		
	FlowEdge(int v, int w, double capacity)	*create a flow edge v→w*
int	from()	*vertex this edge points from*
int	to()	*vertex this edge points to*
int	other(int v)	*other endpoint*
double	capacity()	*capacity of this edge*
double	flow()	*flow in this edge*
double	residualCapacityTo(int v)	*residual capacity toward v*
void	addResidualFlowTo(int v, double delta)	*add delta flow toward v*





Flow edge: Java implementation

```
public class FlowEdge
{
   private final int v, w;          // from and to
   private final double capacity;   // capacity
   private double flow;             // flow

   public FlowEdge(int v, int w, double capacity)
   {
      this.v        = v;
      this.w        = w;
      this.capacity = capacity;
   }

   public int from()              { return v;        }
   public int to()                { return w;        }
   public double capacity()       { return capacity; }
   public double flow()           { return flow;     }

   public int other(int vertex)
   {
      if      (vertex == v) return w;
      else if (vertex == w) return v;
      else throw new IllegalArgumentException();
   }

   public double residualCapacityTo(int vertex)             {...}
   public void addResidualFlowTo(int vertex, double delta)  {...}
}
```

flow variable (mutable)

next slide

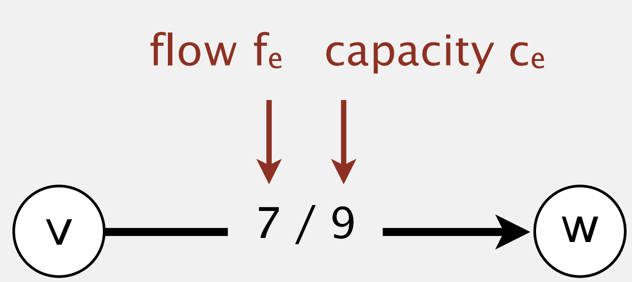
Flow edge: Java implementation (continued)

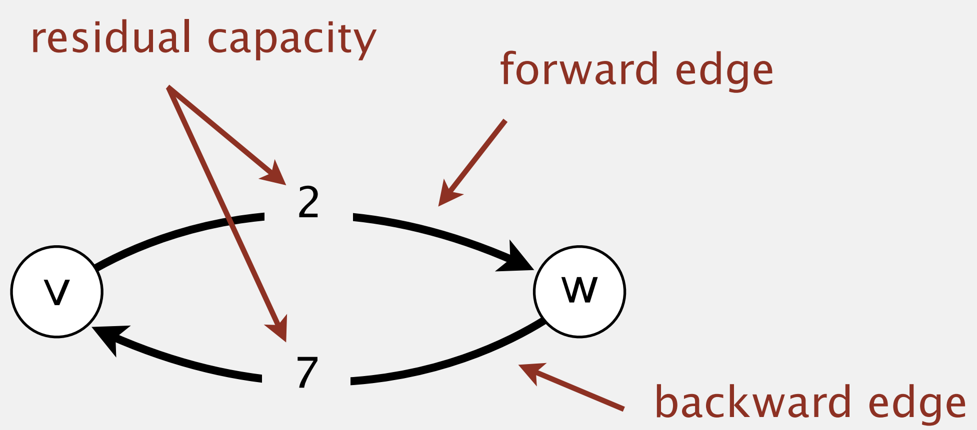
```
public double residualCapacityTo(int vertex)
{
   if      (vertex == v) return flow;
   else if (vertex == w) return capacity - flow;
   else throw new IllegalArgumentException();
}
```

forward edge
backward edge

```
public void addResidualFlowTo(int vertex, double delta)
{
   if      (vertex == v) flow -= delta;
   else if (vertex == w) flow += delta;
   else throw new IllegalArgumentException();
}
```

forward edge
backward edge





Flow network API

public class FlowNetwork		
	`FlowNetwork(int V)`	*create an empty flow network with V vertices*
	`FlowNetwork(In in)`	*construct flow network input stream*
`void`	`addEdge(FlowEdge e)`	*add flow edge e to this flow network*
`Iterable<FlowEdge>`	`adj(int v)`	*forward and backward edges incident to/from v*
`Iterable<FlowEdge>`	`edges()`	*all edges in this flow network*
`int`	`V()`	*number of vertices*
`int`	`E()`	*number of edges*
`String`	`toString()`	*string representation*

Conventions. Allow self-loops and parallel edges.

Flow network: Java implementation

```
public class FlowNetwork
{
    private final int V;
    private Bag<FlowEdge>[] adj;

    public FlowNetwork(int V)
    {
       this.V = V;
       adj = (Bag<FlowEdge>[]) new Bag[V];
       for (int v = 0; v < V; v++)
          adj[v] = new Bag<FlowEdge>();
    }

    public void addEdge(FlowEdge e)
    {
       int v = e.from();
       int w = e.to();
       adj[v].add(e);
       adj[w].add(e);
    }

    public Iterable<FlowEdge> adj(int v)
    {  return adj[v]; }
}
```

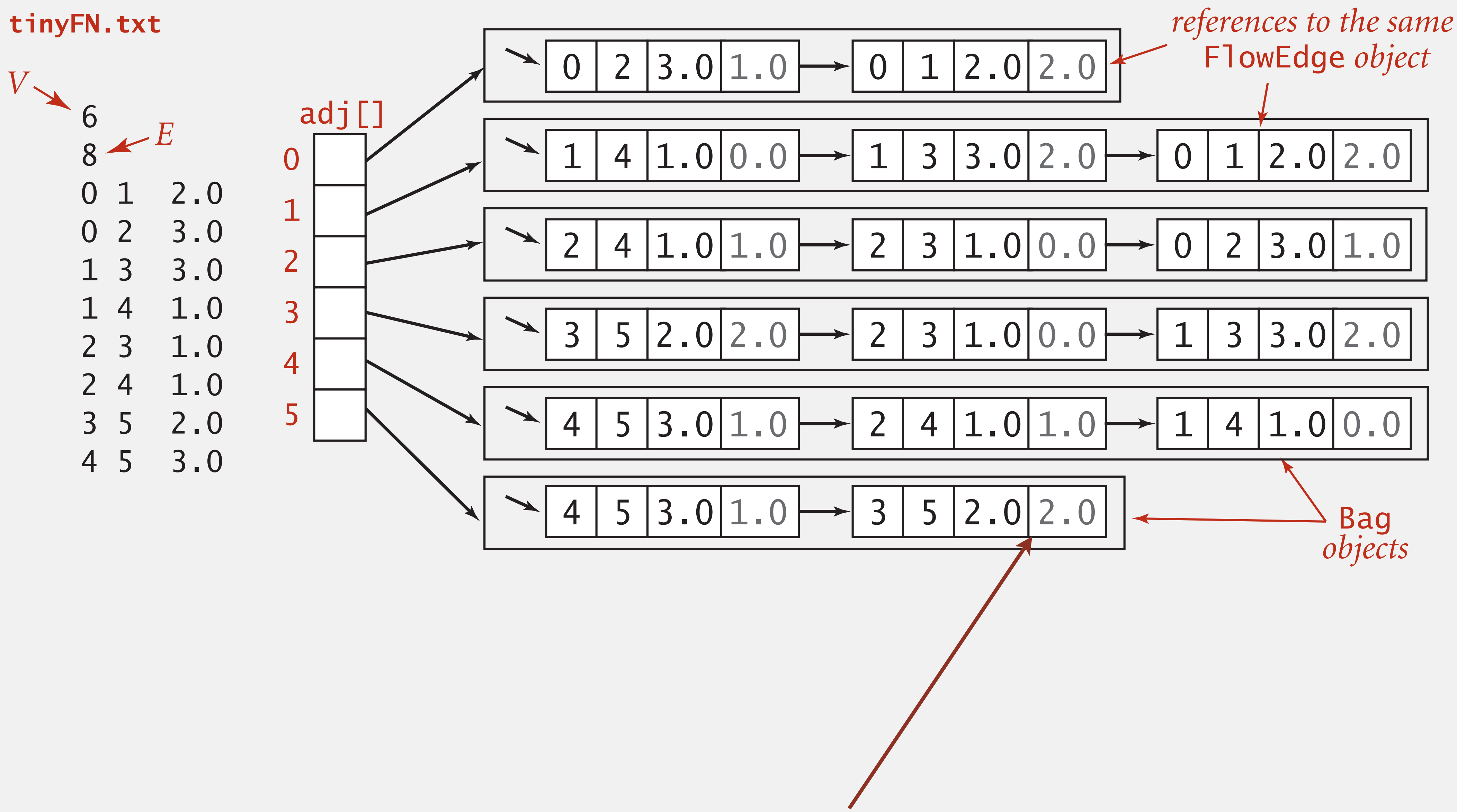
same as EdgeWeightedGraph, but adjacency lists of FlowEdges instead of Edges

add forward edge

add backward edge

Flow network: adjacency-lists representation

Maintain vertex-indexed array of `FlowEdge` lists (use `Bag` abstraction).



Note. Adjacency list includes edges with 0 residual capacity.
(residual network is represented implicitly)

Finding a shortest augmenting path (cf. breadth-first search)

```
private boolean hasAugmentingPath(FlowNetwork G, int s, int t)
{
    edgeTo = new FlowEdge[G.V()];
    marked = new boolean[G.V()];

    Queue<Integer> queue = new Queue<Integer>();
    queue.enqueue(s);
    marked[s] = true;
    while (!queue.isEmpty())
    {
        int v = queue.dequeue();

        for (FlowEdge e : G.adj(v))
        {
            int w = e.other(v);
            if (!marked[w] && (e.residualCapacityTo(w) > 0) )
            {
                edgeTo[w] = e;
                marked[w] = true;
                queue.enqueue(w);
            }
        }
    }

    return marked[t];
}
```

found path from s to w in the residual network?

save last edge on path to w;
mark w;
add w to the queue

is t reachable from s in residual network?

Ford-Fulkerson: Java implementation

```
public class FordFulkerson
{
   private boolean[] marked;    // true if s->v path in residual network
   private FlowEdge[] edgeTo;   // last edge on s->v path
   private double value;        // value of flow

   public FordFulkerson(FlowNetwork G, int s, int t)
   {
      value = 0.0;
      while (hasAugmentingPath(G, s, t))
      {
         double bottle = Double.POSITIVE_INFINITY;
         for (int v = t; v != s; v = edgeTo[v].other(v))
            bottle = Math.min(bottle, edgeTo[v].residualCapacityTo(v));

         for (int v = t; v != s; v = edgeTo[v].other(v))
            edgeTo[v].addResidualFlowTo(v, bottle);

         value += bottle;
      }
   }

   private boolean hasAugmentingPath(FlowNetwork G, int s, int t)
   {  /* See previous slide. */  }

   public double value()
   {  return value;  }

   public boolean inCut(int v)
   {  return marked[v];  }
}
```

compute edgeTo[] and marked[]

compute bottleneck capacity

augment flow

is v reachable from s in residual network?

http://algs4.cs.princeton.edu

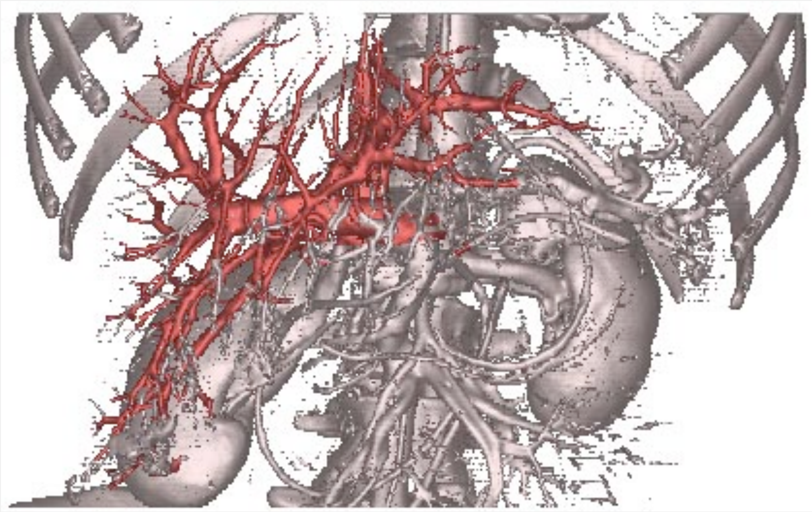
6.4 Maximum Flow

- introduction
- Ford-Fulkerson algorithm
- maxflow-mincut theorem
- analysis of running time
- Java implementation
- *applications*

Maxflow and mincut applications

Maxflow/mincut is a widely applicable problem-solving model.

- Data mining.
- Open-pit mining.
- Bipartite matching.
- Network reliability.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Distributed computing.
- Security of statistical data.
- Egalitarian stable matching.
- Multi-camera scene reconstruction.
- Sensor placement for homeland security.
- Many, many, more.



liver and hepatic vascularization segmentation

Bipartite matching problem

N students apply for N jobs.

Each gets several offers.

Is there a way to match all students to jobs?

bipartite matching problem

1 Alice
- Adobe
- Amazon
- Google

2 Bob
- Adobe
- Amazon

3 Carol
- Adobe
- Facebook
- Google

4 Dave
- Amazon
- Yahoo

5 Eliza
- Amazon
- Yahoo

6 Adobe
- Alice
- Bob
- Carol

7 Amazon
- Alice
- Bob
- Dave
- Eliza

8 Facebook
- Carol

9 Google
- Alice
- Carol

10 Yahoo
- Dave
- Eliza

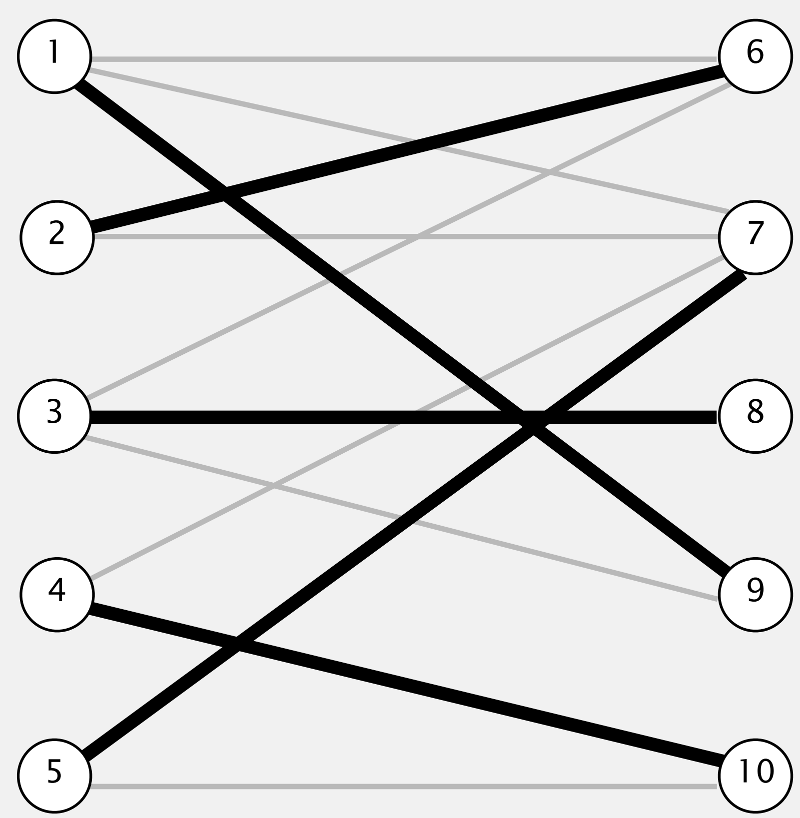
Bipartite matching problem

Given a bipartite graph, find a perfect matching.

perfect matching (solution)

Alice	—	Google
Bob	—	Adobe
Carol	—	Facebook
Dave	—	Yahoo
Eliza	—	Amazon

bipartite graph



N students **N companies**

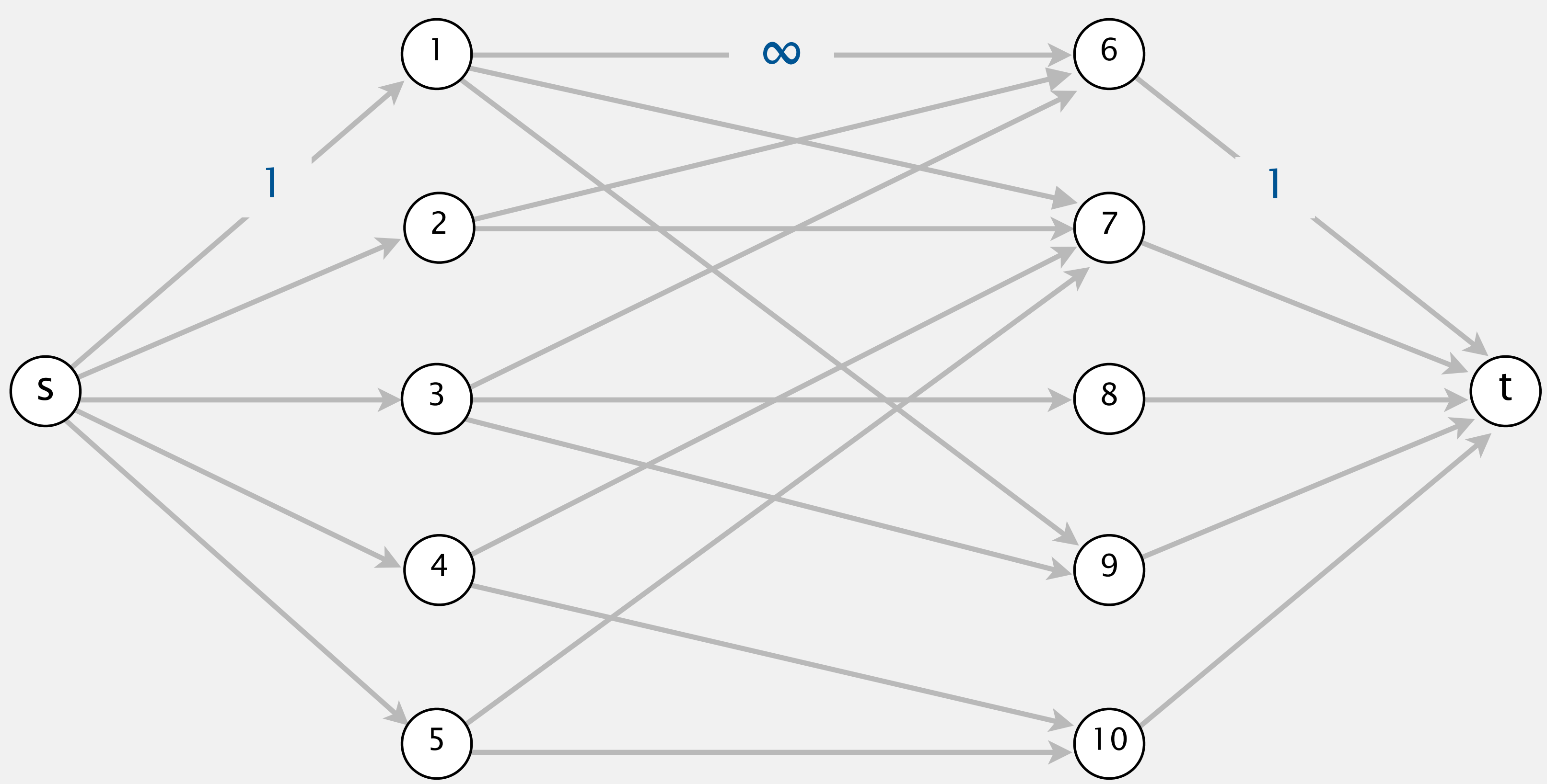
bipartite matching problem

1	Alice	6	Adobe
	Adobe		Alice
	Amazon		Bob
	Google		Carol
2	Bob	7	Amazon
	Adobe		Alice
	Amazon		Bob
3	Carol		Dave
	Adobe		Eliza
	Facebook	8	Facebook
	Google		Carol
4	Dave	9	Google
	Amazon		Alice
	Yahoo		Carol
5	Eliza	10	Yahoo
	Amazon		Dave
	Yahoo		Eliza

Network flow formulation of bipartite matching

- Create s, t, one vertex for each student, and one vertex for each job.
- Add edge from s to each student (capacity 1).
- Add edge from each job to t (capacity 1).
- Add edge from student to each job offered (infinite capacity).

flow network



N students **N companies**

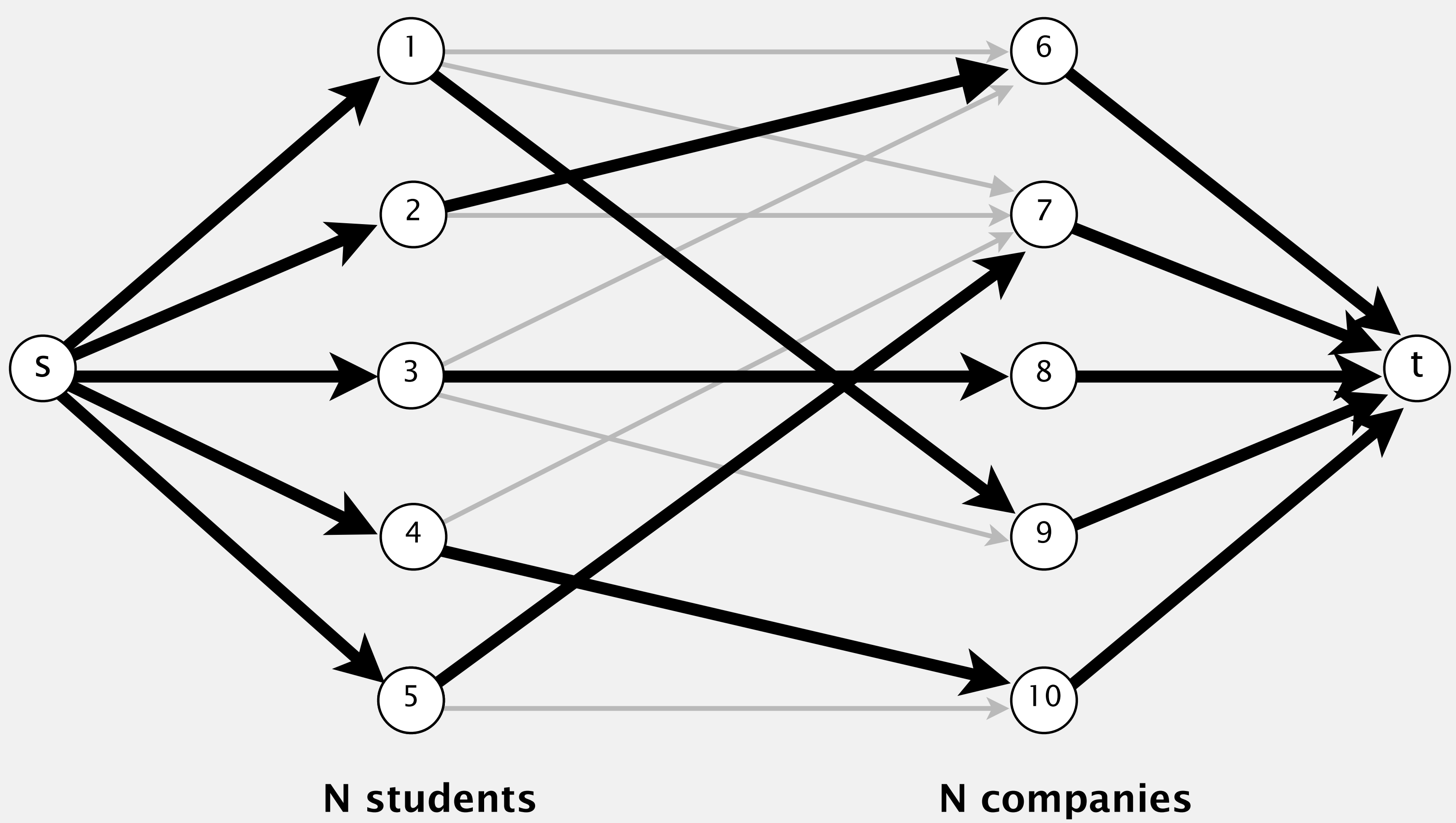
bipartite matching problem

1 Alice	6 Adobe
Adobe	Alice
Amazon	Bob
Google	Carol
2 Bob	7 Amazon
Adobe	Alice
Amazon	Bob
3 Carol	Dave
Adobe	Eliza
Facebook	8 Facebook
Google	Carol
4 Dave	9 Google
Amazon	Alice
Yahoo	Carol
5 Eliza	10 Yahoo
Amazon	Dave
Yahoo	Eliza

Network flow formulation of bipartite matching

1-1 correspondence between perfect matchings in bipartite graph and integer-valued maxflows of value N.

flow network

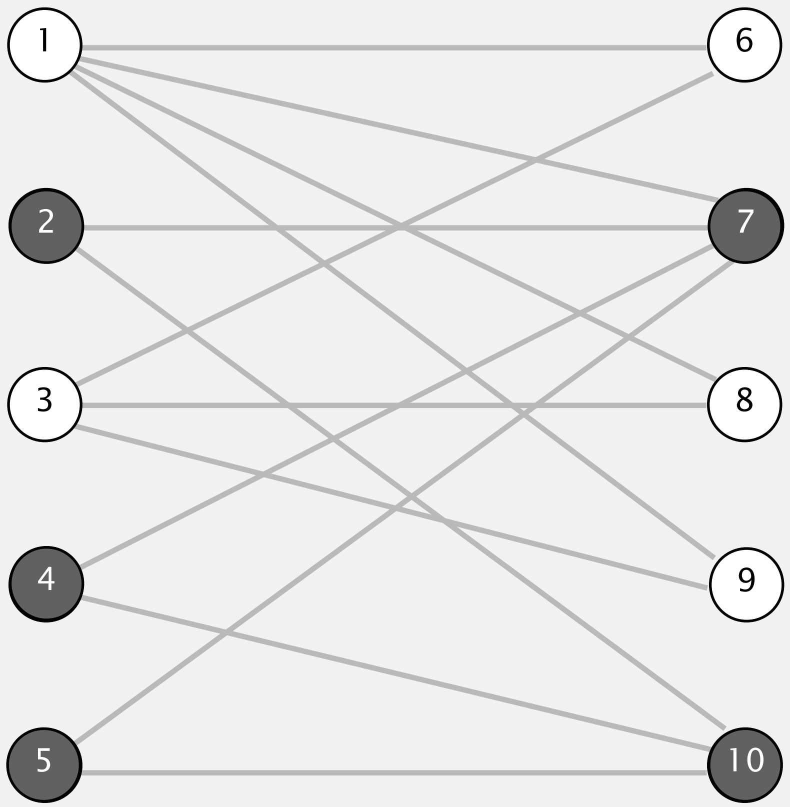


bipartite matching problem

1 Alice	6 Adobe
Adobe	Alice
Amazon	Bob
Google	Carol
2 Bob	7 Amazon
Adobe	Alice
Amazon	Bob
3 Carol	Dave
Adobe	Eliza
Facebook	8 Facebook
Google	Carol
4 Dave	9 Google
Amazon	Alice
Yahoo	Carol
5 Eliza	10 Yahoo
Amazon	Dave
Yahoo	Eliza

What the mincut tells us

Goal. When no perfect matching, explain why.



no perfect matching exists

S = { 2, 4, 5 }
T = { 7, 10 }

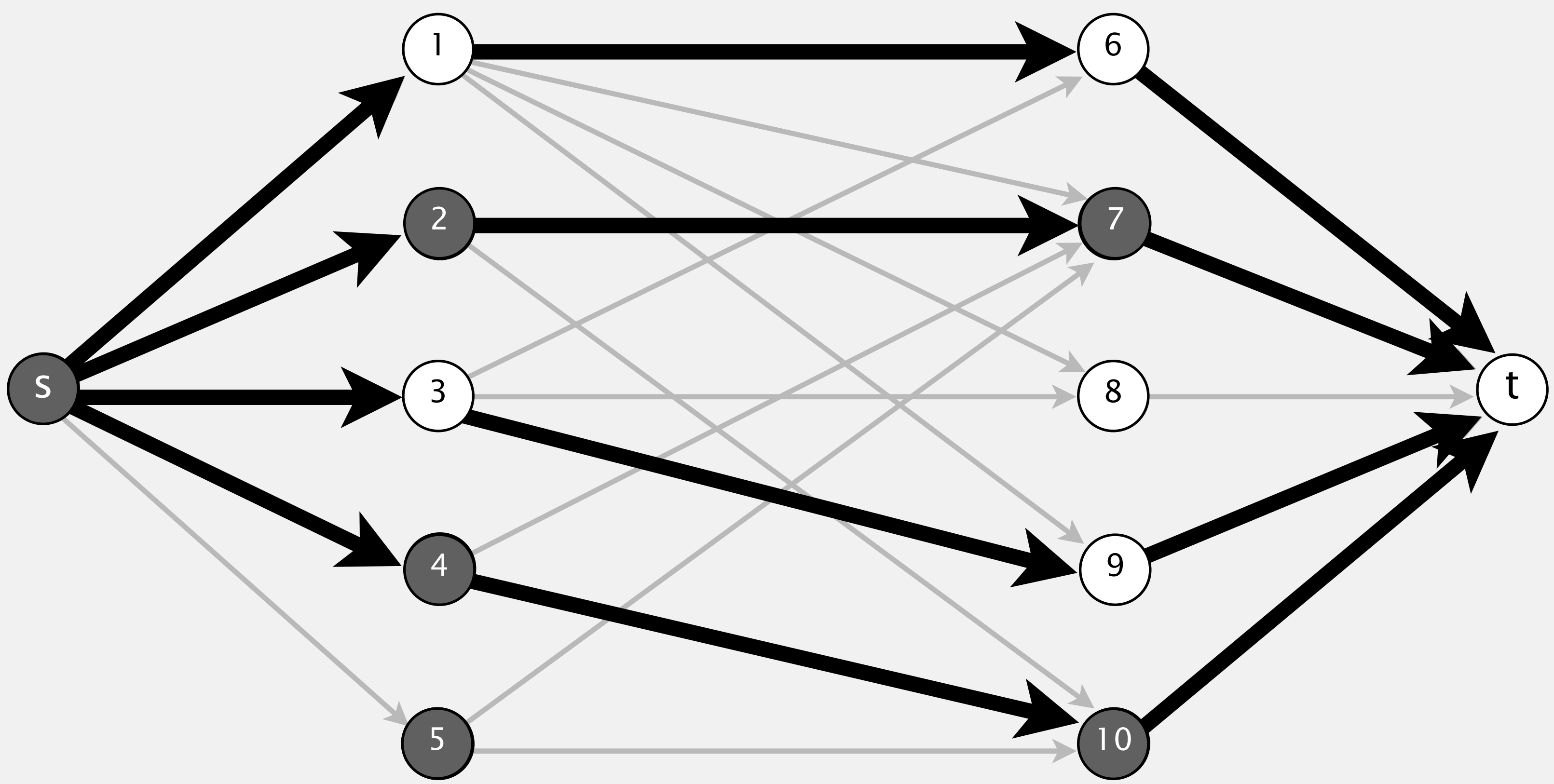
student in S
can be matched
only to
companies in T

| S | > | T |

What the mincut tells us

Mincut. Consider mincut (A, B).

- Let S = students on s side of cut.
- Let T = companies on s side of cut.
- Fact: $|S| > |T|$; students in S can be matched only to companies in T.



no perfect matching exists

S = { 2, 4, 5 }
T = { 7, 10 }

student in S can be matched only to companies in T

| S | > | T |

Bottom line. When no perfect matching, mincut explains why.

Baseball elimination problem

Q. Which teams have a chance of finishing the season with the most wins?

i	team		wins	losses	to play	ATL	PHI	NYM	WAS
0		Atlanta	83	71	8	–	1	6	1
1		Philly	80	79	3	1	–	0	2
2		New York	78	78	6	6	0	–	0
3		Washington	77	82	3	1	2	0	–

Washington is mathematically eliminated.

- Washington finishes with ≤ 80 wins.
- Atlanta already has 83 wins.

Baseball elimination problem

Q. Which teams have a chance of finishing the season with the most wins?

i	team		wins	losses	to play	ATL	PHI	NYM	WAS
0		Atlanta	83	71	8	–	1	6	1
1		Philly	80	79	3	1	–	0	2
2		New York	78	78	6	6	0	–	0
3		Washington	77	82	3	1	2	0	–

Philadelphia is mathematically eliminated.

- Philadelphia finishes with ≤ 83 wins.
- Either New York or Atlanta will finish with ≥ 84 wins.

Observation. Answer depends not only on how many games already won and left to play, but on whom they're against.

Baseball elimination problem

Q. Which teams have a chance of finishing the season with the most wins?

i	team	wins	losses	to play	NYY	BAL	BOS	TOR	DET
0	New York	75	59	28	–	3	8	7	3
1	Baltimore	71	63	28	3	–	2	7	4
2	Boston	69	66	27	8	2	–	0	0
3	Toronto	63	72	27	7	7	0	–	0
4	Detroit	49	86	27	3	4	0	0	–

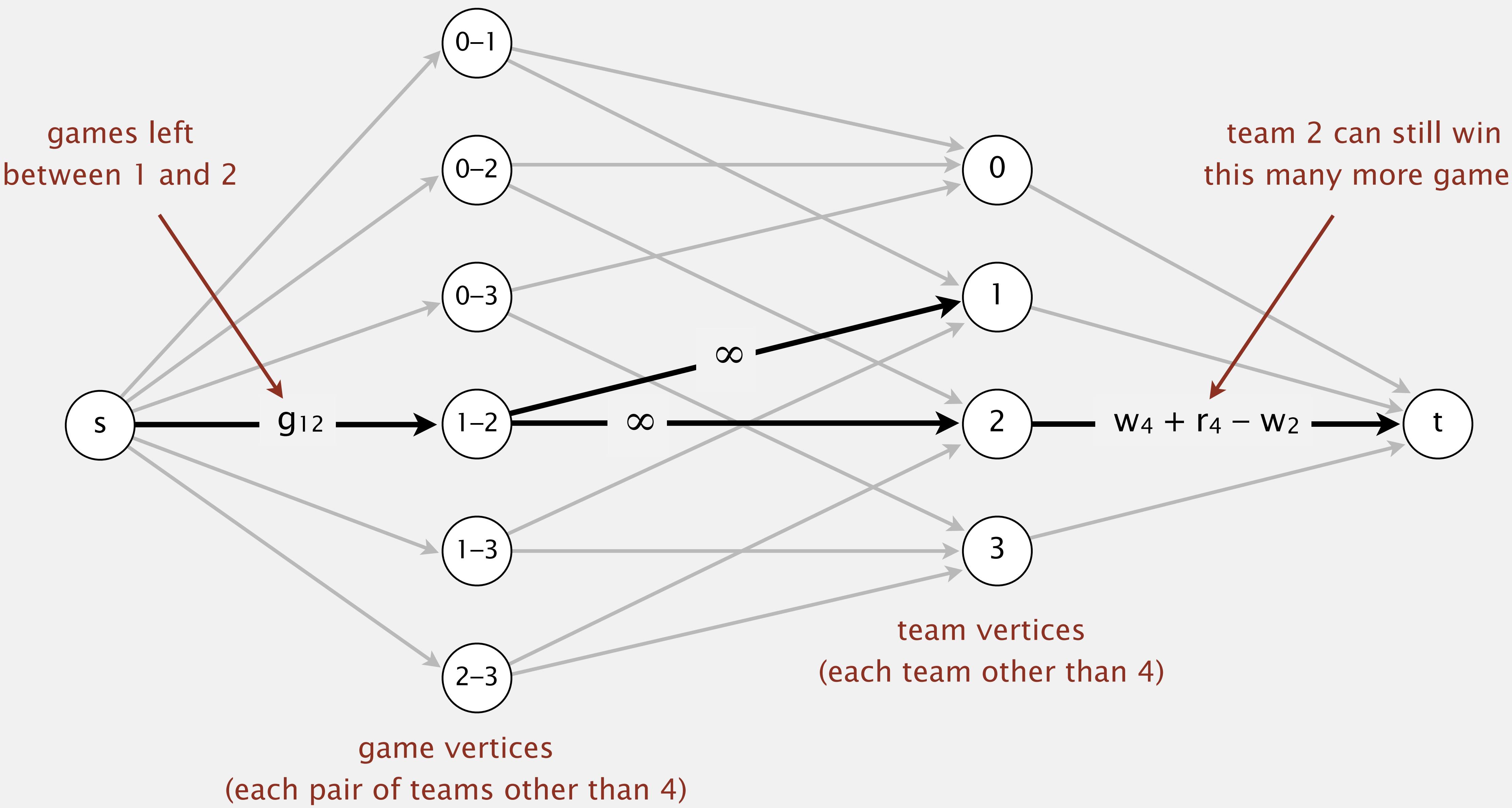
AL East (August 30, 1996)

Detroit is mathematically eliminated.

- Detroit finishes with ≤ 76 wins.
- Wins for $R = \{$ NYY, BAL, BOS, TOR $\} = 278$.
- Remaining games among $\{$ NYY, BAL, BOS, TOR $\} = 3 + 8 + 7 + 2 + 7 = 27$.
- Average team in R wins $305/4 = 76.25$ games.

Baseball elimination problem: maxflow formulation

Intuition. Remaining games flow from s to t.



Fact. Team 4 not eliminated iff all edges pointing from s are full in maxflow.

Maximum flow algorithms: theory

(Yet another) holy grail for theoretical computer scientists.

year	method	worst case	discovered by
1951	**simplex**	$E^3 U$	Dantzig
1955	**augmenting path**	$E^2 U$	Ford-Fulkerson
1970	**shortest augmenting path**	E^3	Dinitz, Edmonds-Karp
1970	**fattest augmenting path**	$E^2 \log E \log(EU)$	Dinitz, Edmonds-Karp
1977	**blocking flow**	$E^{5/2}$	Cherkasky
1978	**blocking flow**	$E^{7/3}$	Galil
1983	**dynamic trees**	$E^2 \log E$	Sleator-Tarjan
1985	**capacity scaling**	$E^2 \log U$	Gabow
1997	**length function**	$E^{3/2} \log E \log U$	Goldberg-Rao
2012	**compact network**	$E^2 / \log E$	Orlin
?	**?**	E	?

maxflow algorithms for sparse networks with E edges, integer capacities between 1 and U

Maximum flow algorithms: practice

Warning. Worst-case order-of-growth is generally not useful for predicting or comparing maxflow algorithm performance in practice.

Best in practice. Push-relabel method with gap relabeling: $E^{3/2}$.

On Implementing Push-Relabel Method for the Maximum Flow Problem

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Abstract. We study efficient implementations of the push-relabel method for the maximum flow problem. The resulting codes are faster than the previous codes, and much faster on some problem families. The speedup is due to the combination of heuristics used in our implementations. We also exhibit a family of problems for which the running time of all known methods seem to have a roughly quadratic growth rate.

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European Journal of Operational Research 97 (1997) 509–542

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Theory and Methodology

Computational investigations of maximum flow algorithms

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Summary

Mincut problem. Find an *st*-cut of minimum capacity.

Maxflow problem. Find an *st*-flow of maximum value.

Duality. Value of the maxflow = capacity of mincut.

Proven successful approaches.

- Ford-Fulkerson (various augmenting-path strategies).
- Preflow-push (various versions).

Open research challenges.

- Practice: solve real-world maxflow/mincut problems in linear time.
- Theory: prove it for worst-case inputs.
- Still much to be learned!