



<http://algs4.cs.princeton.edu>

## 4.4 SHORTEST PATHS

---

- ▶ *APIs*
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ *edge-weighted DAGs*
- ▶ *negative weights*

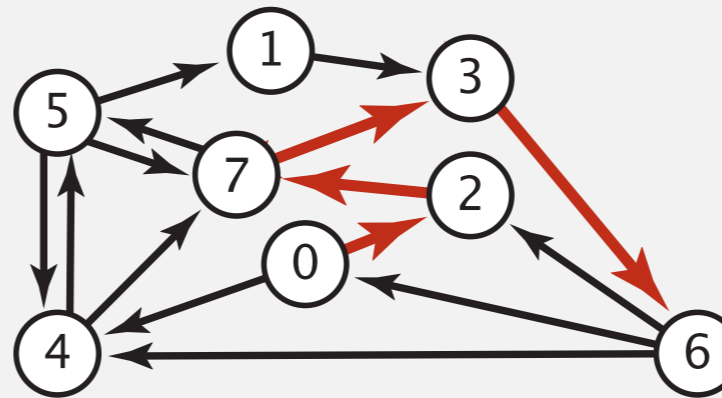
# Shortest paths in an edge-weighted digraph

---

Given an edge-weighted digraph, find the shortest path from  $s$  to  $t$ .

## edge-weighted digraph

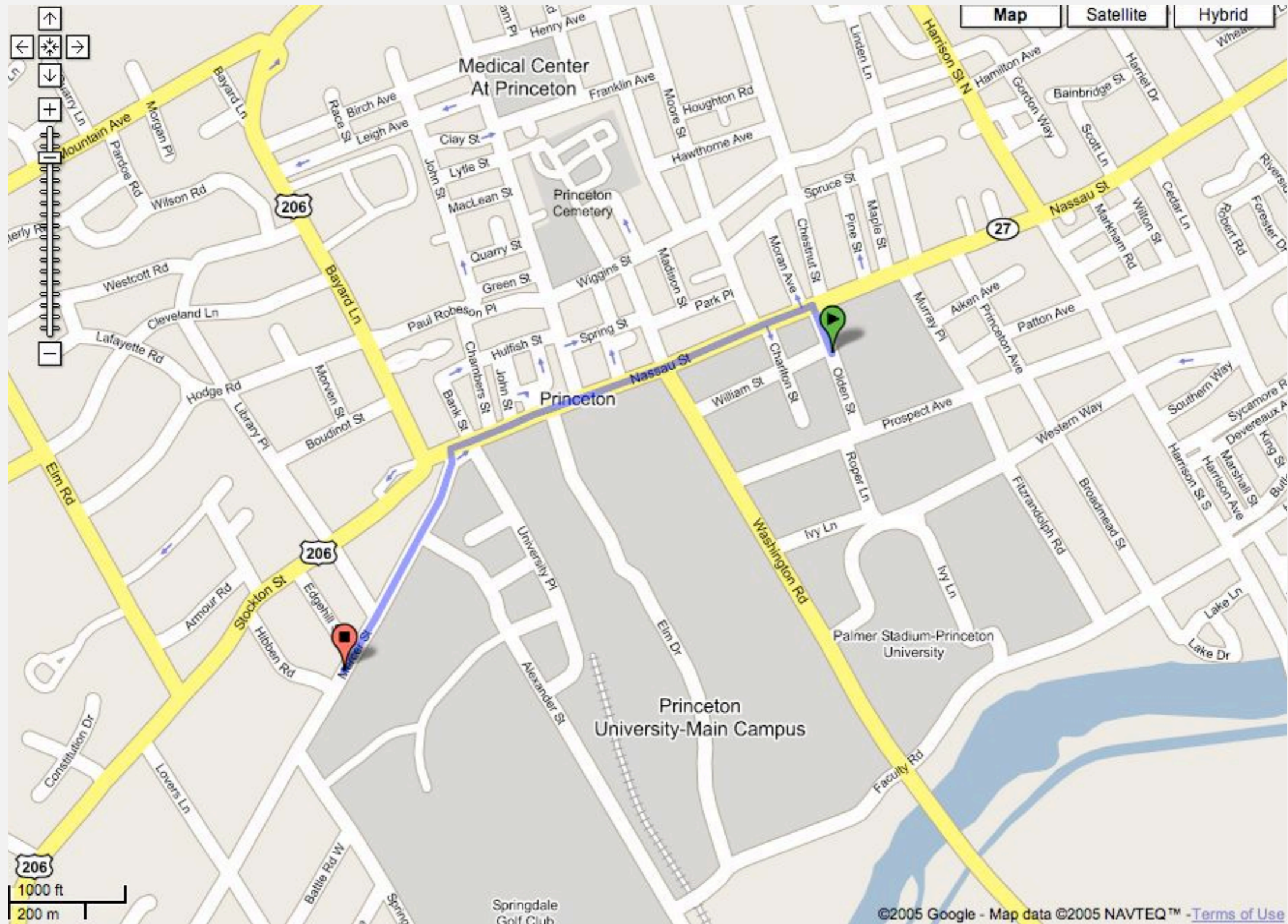
4→5	0.35
5→4	0.35
4→7	0.37
5→7	0.28
7→5	0.28
5→1	0.32
0→4	0.38
0→2	0.26
7→3	0.39
1→3	0.29
2→7	0.34
6→2	0.40
3→6	0.52
6→0	0.58
6→4	0.93



## shortest path from 0 to 6

0→2	0.26
2→7	0.34
7→3	0.39
3→6	0.52

# Google maps





# Shortest path applications

---

- PERT/CPM.
- Map routing.
- **Seam carving.**
- Texture mapping.
- Robot navigation.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting **arbitrage** opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.



[http://en.wikipedia.org/wiki/Seam\\_carving](http://en.wikipedia.org/wiki/Seam_carving)



Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

# Shortest path variants

---

## Which vertices?

- **Single source:** from one vertex  $s$  to every other vertex.
- Single sink: from every vertex to one vertex  $t$ .
- Source-sink: from one vertex  $s$  to another  $t$ .
- All pairs: between all pairs of vertices.

## Restrictions on edge weights?

- Nonnegative weights.
- Euclidean weights.
- Arbitrary weights.

## Cycles?

- No directed cycles.
- No "negative cycles."



which variant?

**Simplifying assumption.** Shortest paths from  $s$  to each vertex  $v$  exist.



# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 4.4 SHORTEST PATHS

---

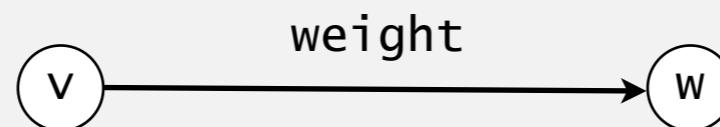
- ▶ *APIs*
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ *edge-weighted DAGs*
- ▶ *negative weights*

# Weighted directed edge API

---

```
public class DirectedEdge
```

```
    DirectedEdge(int v, int w, double weight)    weighted edge v→w  
    int from()                                  vertex v  
    int to()                                    vertex w  
    double weight()                             weight of this edge  
    String toString()                           string representation
```



Idiom for processing an edge `e`: `int v = e.from(), w = e.to();`

# Weighted directed edge: implementation in Java

---

Similar to Edge for undirected graphs, but a bit simpler.

```
public class DirectedEdge
{
    private final int v, w;
    private final double weight;

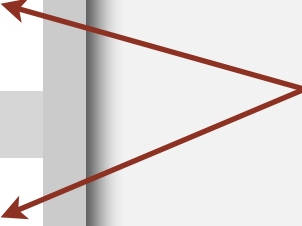
    public DirectedEdge(int v, int w, double weight)
    {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }
}
```

```
public int from()
{ return v; }
```

```
public int to()
{ return w; }
```

```
public int weight()
{ return weight; }
```

```
}
```



from() and to() replace  
either() and other()



# Edge-weighted digraph API

---

```
public class EdgeWeightedDigraph
```

```
    EdgeWeightedDigraph(int V) edge-weighted digraph with V vertices
```

```
    EdgeWeightedDigraph(In in) edge-weighted digraph from input stream
```

```
    void addEdge(DirectedEdge e) add weighted directed edge e
```

```
    Iterable<DirectedEdge> adj(int v) edges adjacent from v
```

```
    int V() number of vertices
```

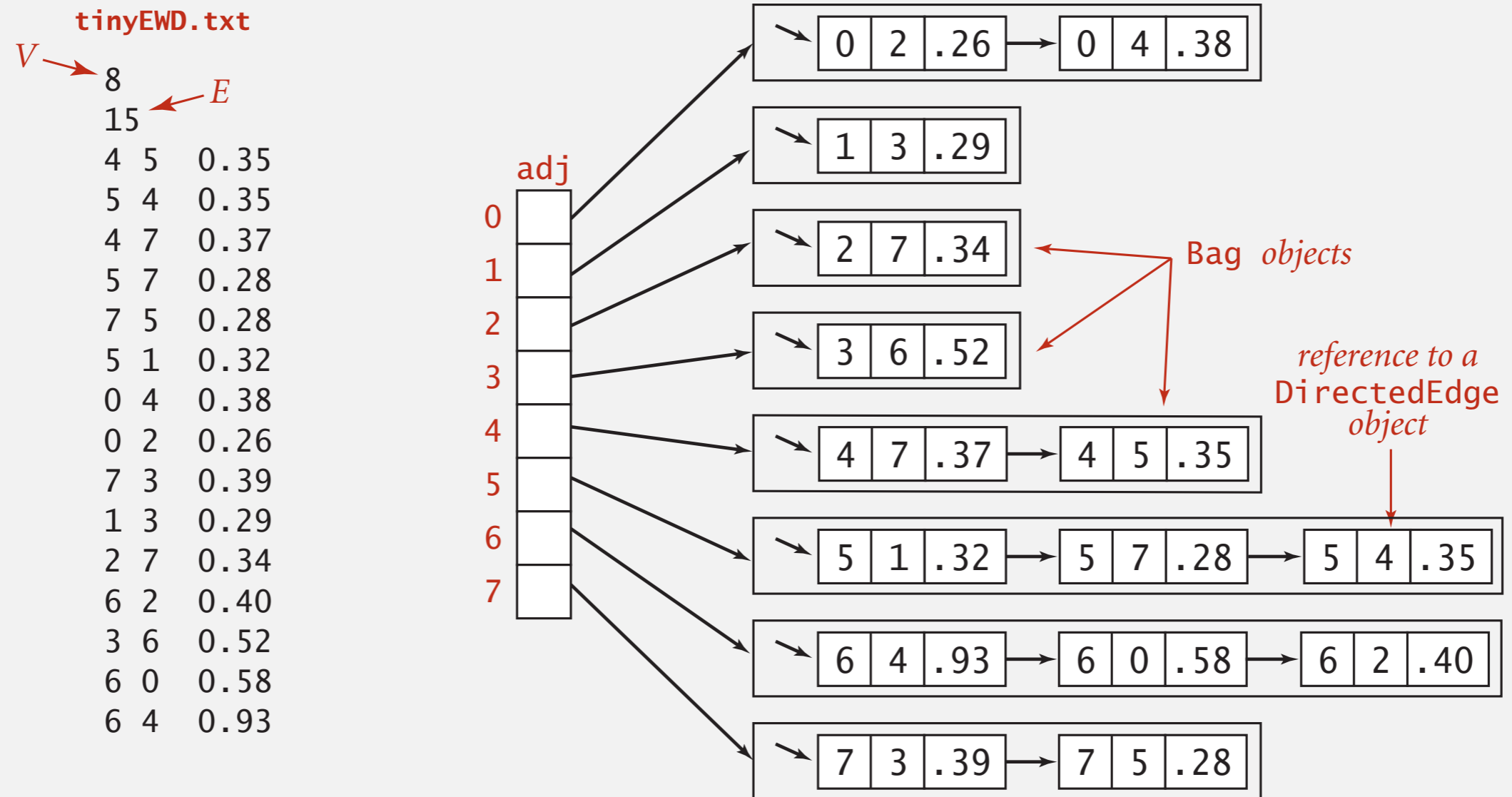
```
    int E() number of edges
```

```
    Iterable<DirectedEdge> edges() all edges
```

```
    String toString() string representation
```

**Conventions.** Allow self-loops and parallel edges.

# Edge-weighted digraph: adjacency-lists representation



# Edge-weighted digraph: adjacency-lists implementation in Java

---

Same as EdgeWeightedGraph except replace Graph with Digraph.

```
public class EdgeWeightedDigraph
{
    private final int V;
    private final Bag<DirectedEdge>[] adj;

    public EdgeWeightedDigraph(int V)
    {
        this.V = V;
        adj = (Bag<DirectedEdge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<DirectedEdge>();
    }

    public void addEdge(DirectedEdge e)
    {
        int v = e.from();
        adj[v].add(e);
    }

    public Iterable<DirectedEdge> adj(int v)
    { return adj[v]; }
}
```

← add edge  $e = v \rightarrow w$  to  
only  $v$ 's adjacency list

# Single-source shortest paths API

---

**Goal.** Find the shortest path from  $s$  to every other vertex.

```
public class SP
```

```
    SP(EdgeWeightedDigraph G, int s) shortest paths from s in graph G
```

```
    double distTo(int v) length of shortest path from s to v
```

```
    Iterable <DirectedEdge> pathTo(int v) shortest path from s to v
```

```
    boolean hasPathTo(int v) is there a path from s to v?
```

```
SP sp = new SP(G, s);
for (int v = 0; v < G.V(); v++)
{
    StdOut.printf("%d to %d (%.2f): ", s, v, sp.distTo(v));
    for (DirectedEdge e : sp.pathTo(v))
        StdOut.print(e + " ");
    StdOut.println();
}
```



# Single-source shortest paths API

---

**Goal.** Find the shortest path from  $s$  to every other vertex.

```
public class SP
```

```
    SP(EdgeWeightedDigraph G, int s) shortest paths from s in graph G
```

```
    double distTo(int v) length of shortest path from s to v
```

```
    Iterable <DirectedEdge> pathTo(int v) shortest path from s to v
```

```
    boolean hasPathTo(int v) is there a path from s to v?
```

```
% java SP tinyEWD.txt 0
0 to 0 (0.00):
0 to 1 (1.05): 0->4 0.38 4->5 0.35 5->1 0.32
0 to 2 (0.26): 0->2 0.26
0 to 3 (0.99): 0->2 0.26 2->7 0.34 7->3 0.39
0 to 4 (0.38): 0->4 0.38
0 to 5 (0.73): 0->4 0.38 4->5 0.35
0 to 6 (1.51): 0->2 0.26 2->7 0.34 7->3 0.39 3->6 0.52
0 to 7 (0.60): 0->2 0.26 2->7 0.34
```



# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 4.4 SHORTEST PATHS

---

- ▶ *APIs*
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ *edge-weighted DAGs*
- ▶ *negative weights*

# Data structures for single-source shortest paths

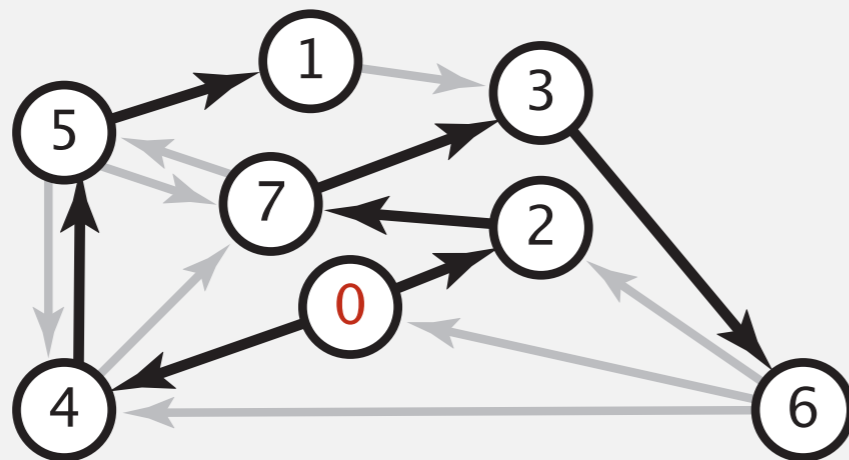
---

**Goal.** Find the shortest path from  $s$  to every other vertex.

**Observation.** A **shortest-paths tree** (SPT) solution exists. Why?

**Consequence.** Can represent the SPT with two vertex-indexed arrays:

- $\text{distTo}[v]$  is length of shortest path from  $s$  to  $v$ .
- $\text{edgeTo}[v]$  is last edge on shortest path from  $s$  to  $v$ .



shortest-paths tree from 0

	<u>edgeTo[]</u>	<u>distTo[]</u>
0	null	0
1	5->1 0.32	1.05
2	0->2 0.26	0.26
3	7->3 0.37	0.97
4	0->4 0.38	0.38
5	4->5 0.35	0.73
6	3->6 0.52	1.49
7	2->7 0.34	0.60

parent-link representation

# Data structures for single-source shortest paths

---

**Goal.** Find the shortest path from  $s$  to every other vertex.

**Observation.** A **shortest-paths tree** (SPT) solution exists. Why?

**Consequence.** Can represent the SPT with two vertex-indexed arrays:

- `distTo[v]` is length of shortest path from  $s$  to  $v$ .
- `edgeTo[v]` is last edge on shortest path from  $s$  to  $v$ .

```
public double distTo(int v)
{ return distTo[v]; }

public Iterable<DirectedEdge> pathTo(int v)
{
    Stack<DirectedEdge> path = new Stack<DirectedEdge>();
    for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
        path.push(e);
    return path;
}
```

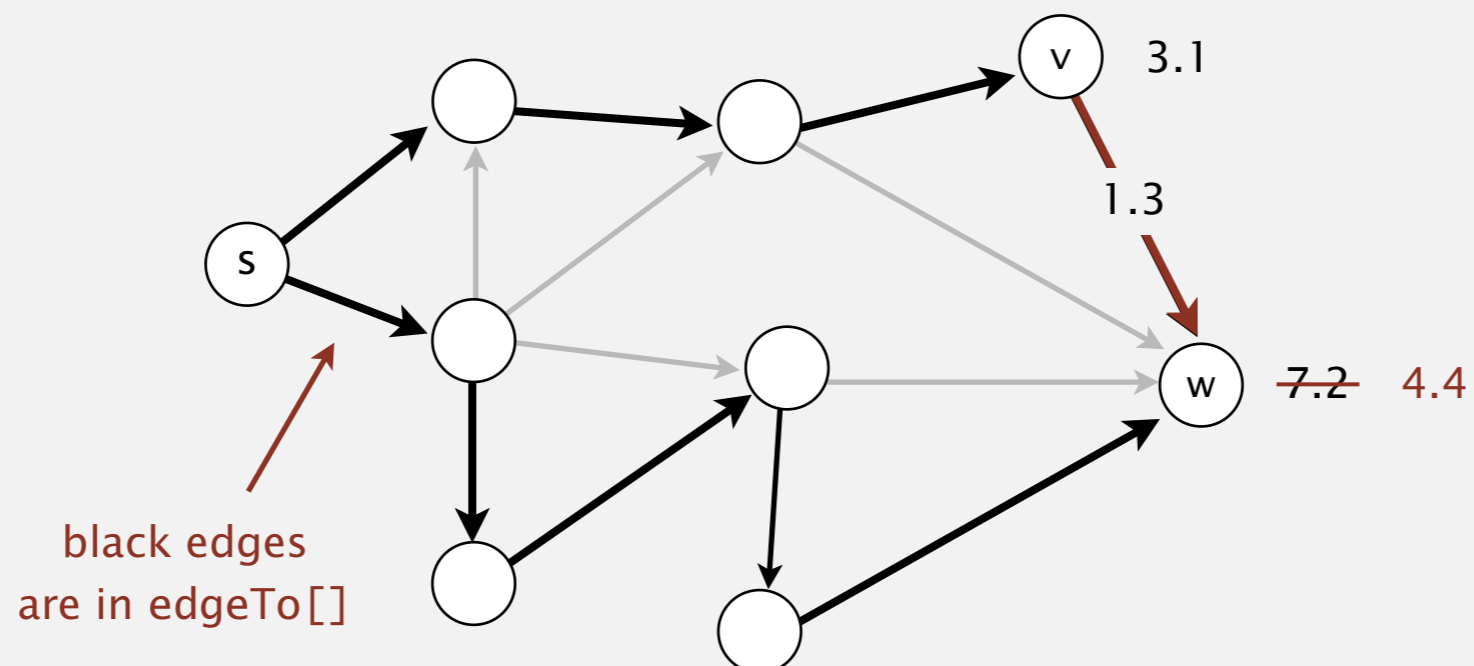


# Edge relaxation

Relax edge  $e = v \rightarrow w$ .

- $\text{distTo}[v]$  is length of shortest **known** path from  $s$  to  $v$ .
- $\text{distTo}[w]$  is length of shortest **known** path from  $s$  to  $w$ .
- $\text{edgeTo}[w]$  is last edge on shortest **known** path from  $s$  to  $w$ .
- If  $e = v \rightarrow w$  gives shorter path to  $w$  through  $v$ , update both  $\text{distTo}[w]$  and  $\text{edgeTo}[w]$ .

$v \rightarrow w$  successfully relaxes



# Edge relaxation

---

Relax edge  $e = v \rightarrow w$ .

- `distTo[v]` is length of shortest **known** path from `s` to `v`.
- `distTo[w]` is length of shortest **known** path from `s` to `w`.
- `edgeTo[w]` is last edge on shortest **known** path from `s` to `w`.
- If  $e = v \rightarrow w$  gives shorter path to `w` through `v`, update both `distTo[w]` and `edgeTo[w]`.

```
private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
    }
}
```

# Shortest-paths optimality conditions

---

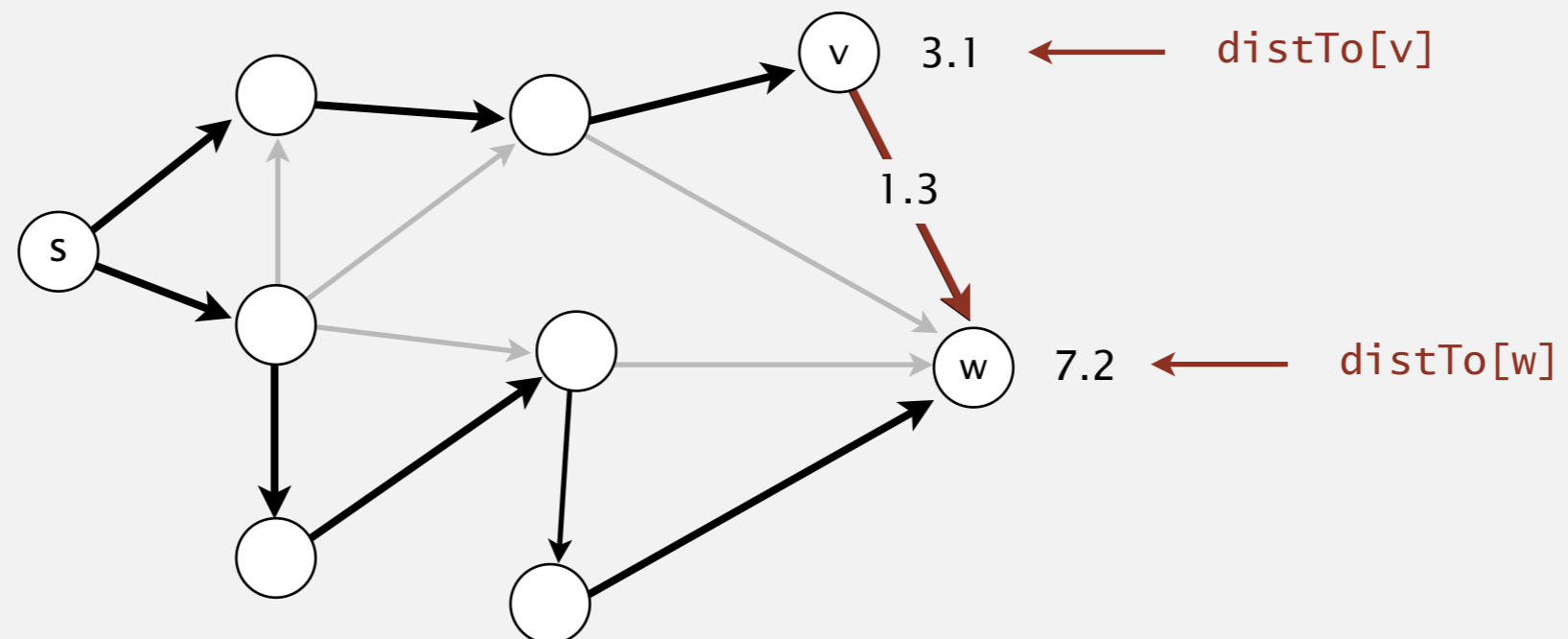
**Proposition.** Let  $G$  be an edge-weighted digraph.

Then  $\text{distTo}[\ ]$  are the shortest path distances from  $s$  iff:

- $\text{distTo}[s] = 0$ .
- For each vertex  $v$ ,  $\text{distTo}[v]$  is the length of some path from  $s$  to  $v$ .
- For each edge  $e = v \rightarrow w$ ,  $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$ .

**Pf.**  $\Leftarrow$  [ necessary ]

- Suppose that  $\text{distTo}[w] > \text{distTo}[v] + e.\text{weight}()$  for some edge  $e = v \rightarrow w$ .
- Then,  $e$  gives a path from  $s$  to  $w$  (through  $v$ ) of length less than  $\text{distTo}[w]$ .



# Shortest-paths optimality conditions

---

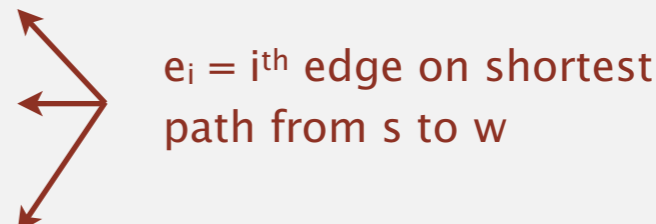
**Proposition.** Let  $G$  be an edge-weighted digraph.

Then  $\text{distTo}[\ ]$  are the shortest path distances from  $s$  iff:

- $\text{distTo}[s] = 0$ .
- For each vertex  $v$ ,  $\text{distTo}[v]$  is the length of some path from  $s$  to  $v$ .
- For each edge  $e = v \rightarrow w$ ,  $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$ .

**Pf.**  $\Rightarrow$  [ sufficient ]

- Suppose that  $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k = w$  is a shortest path from  $s$  to  $w$ .


- Then,  
 $\text{distTo}[v_1] \leq \text{distTo}[v_0] + e_1.\text{weight}()$   
 $\text{distTo}[v_2] \leq \text{distTo}[v_1] + e_2.\text{weight}()$   
...  
 $\text{distTo}[v_k] \leq \text{distTo}[v_{k-1}] + e_k.\text{weight}()$
- 

- Add inequalities; simplify; and substitute  $\text{distTo}[v_0] = \text{distTo}[s] = 0$ :

$$\text{distTo}[w] = \text{distTo}[v_k] \leq \underline{e_1.\text{weight}() + e_2.\text{weight}() + \dots + e_k.\text{weight}()}$$

weight of shortest path from  $s$  to  $w$

- Thus,  $\text{distTo}[w]$  is the weight of shortest path to  $w$ . ■



weight of some path from  $s$  to  $w$



# Generic shortest-paths algorithm

---

## Generic algorithm (to compute SPT from $s$ )

---

Initialize  $\text{distTo}[s] = 0$  and  $\text{distTo}[v] = \infty$  for all other vertices.

Repeat until optimality conditions are satisfied:

- Relax any edge.
- 

**Proposition.** Generic algorithm computes SPT (if it exists) from  $s$ .

**Pf sketch.**

- The entry  $\text{distTo}[v]$  is always the length of a simple path from  $s$  to  $v$ .
- Each successful relaxation decreases  $\text{distTo}[v]$  for some  $v$ .
- The entry  $\text{distTo}[v]$  can decrease at most a finite number of times. ■

# Generic shortest-paths algorithm

---

## Generic algorithm (to compute SPT from $s$ )

---

Initialize  $\text{distTo}[s] = 0$  and  $\text{distTo}[v] = \infty$  for all other vertices.

Repeat until optimality conditions are satisfied:

- Relax any edge.
- 

**Efficient implementations.** How to choose which edge to relax?

**Ex 1.** Dijkstra's algorithm (nonnegative weights).

**Ex 2.** Topological sort algorithm (no directed cycles).

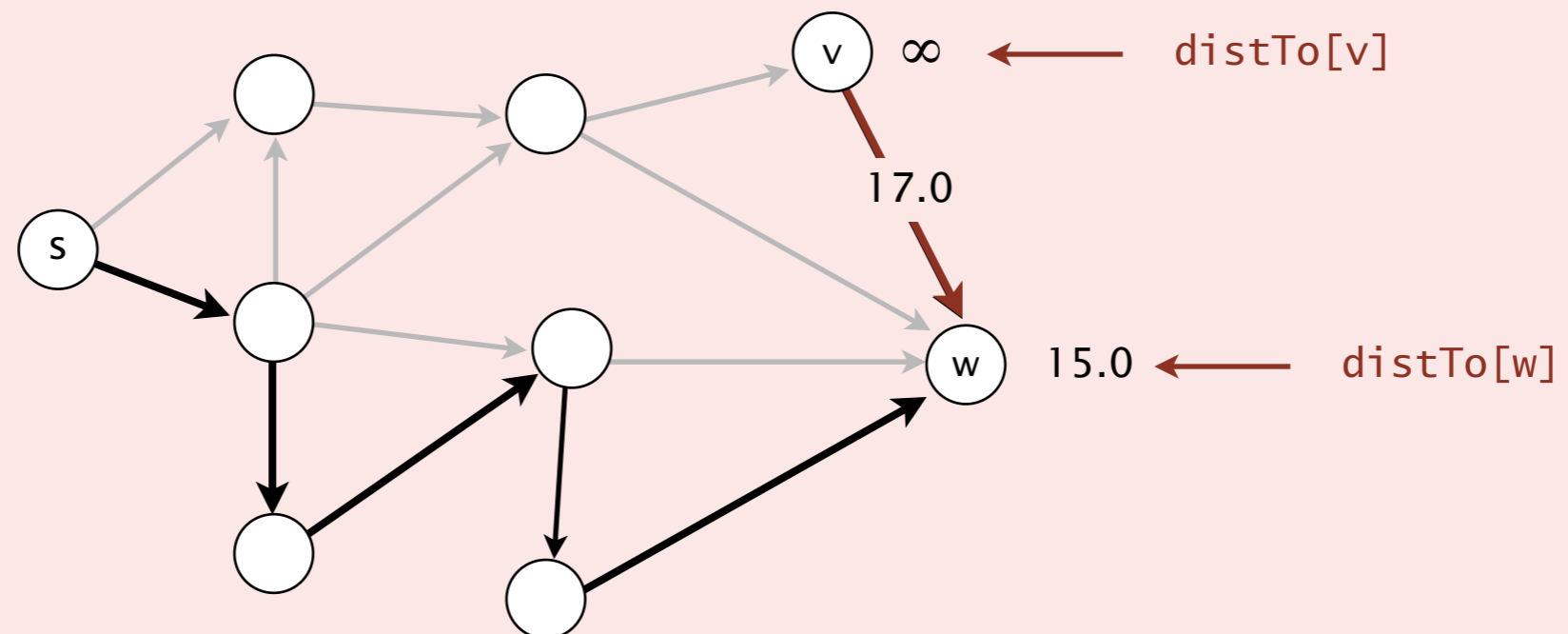
**Ex 3.** Bellman-Ford algorithm (no negative cycles).

# Shortest paths: quiz 1

---

Let  $e = v \rightarrow w$  be an edge with weight 17.0. Suppose that during the generic shortest paths algorithm,  $\text{distTo}[v] = \infty$  and  $\text{distTo}[w] = 15.0$ . What will  $\text{distTo}[w]$  be after calling  $\text{relax}(e)$ ?

- A. *The program will throw a java.lang.RuntimeException.*
- B. 15.0
- C. 17.0
- D.  $+\infty$
- E. *I don't know.*





# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 4.4 SHORTEST PATHS

---

- ▶ *APIs*
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ *edge-weighted DAGs*
- ▶ *negative weights*

# Edsger W. Dijkstra: select quotes

---

*“ Do only what only you can do. ”*

*“ In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind. ”*

*“ The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence. ”*

*“ It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration. ”*

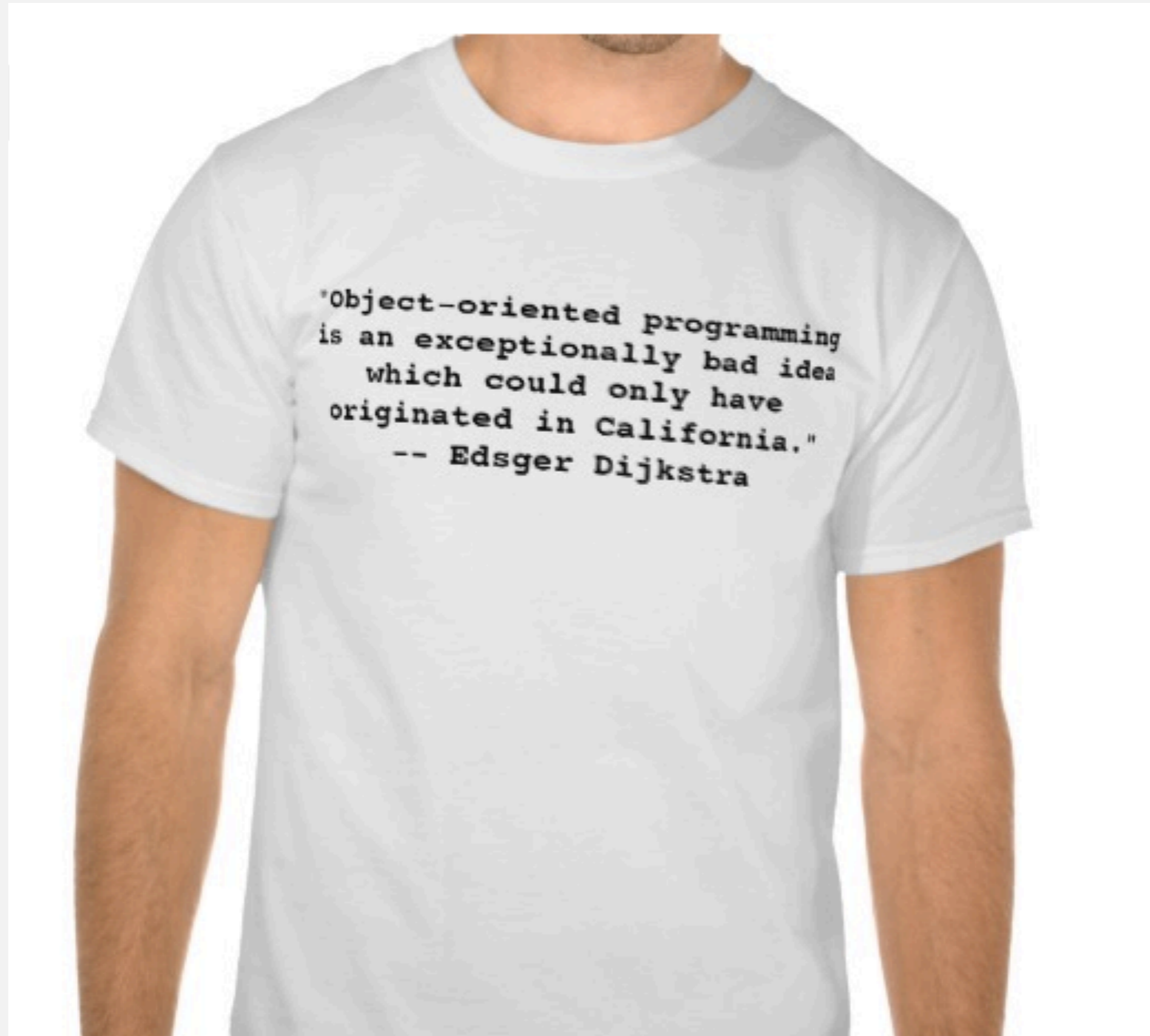
*“ APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums. ”*



**Edsger W. Dijkstra**  
**Turing award 1972**

# Edsger W. Dijkstra: select quotes

---

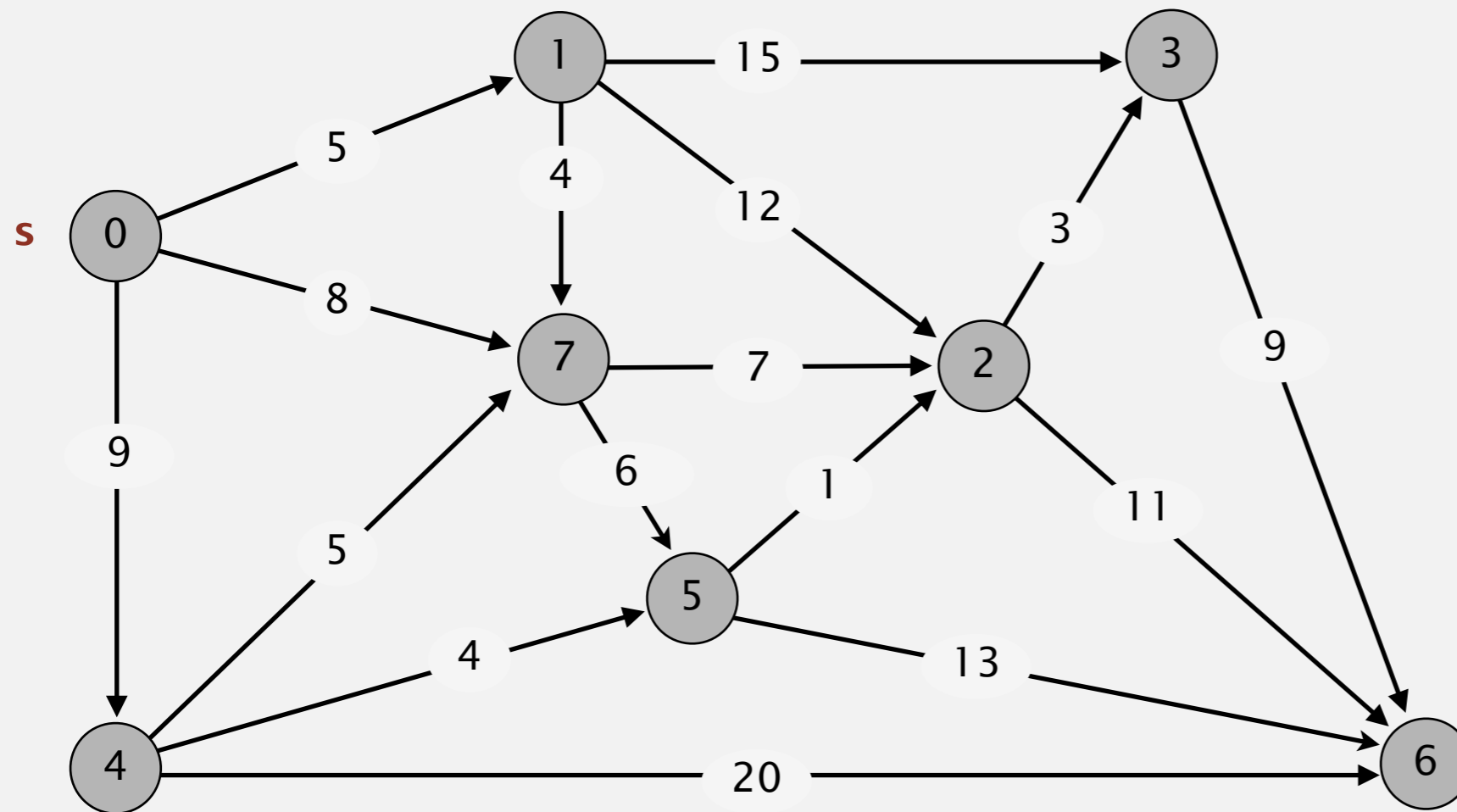




# Dijkstra's algorithm demo



- Consider vertices in increasing order of distance from  $s$  (non-tree vertex with the lowest  $\text{distTo}[]$  value).
- Add vertex to tree and relax all edges adjacent from that vertex.

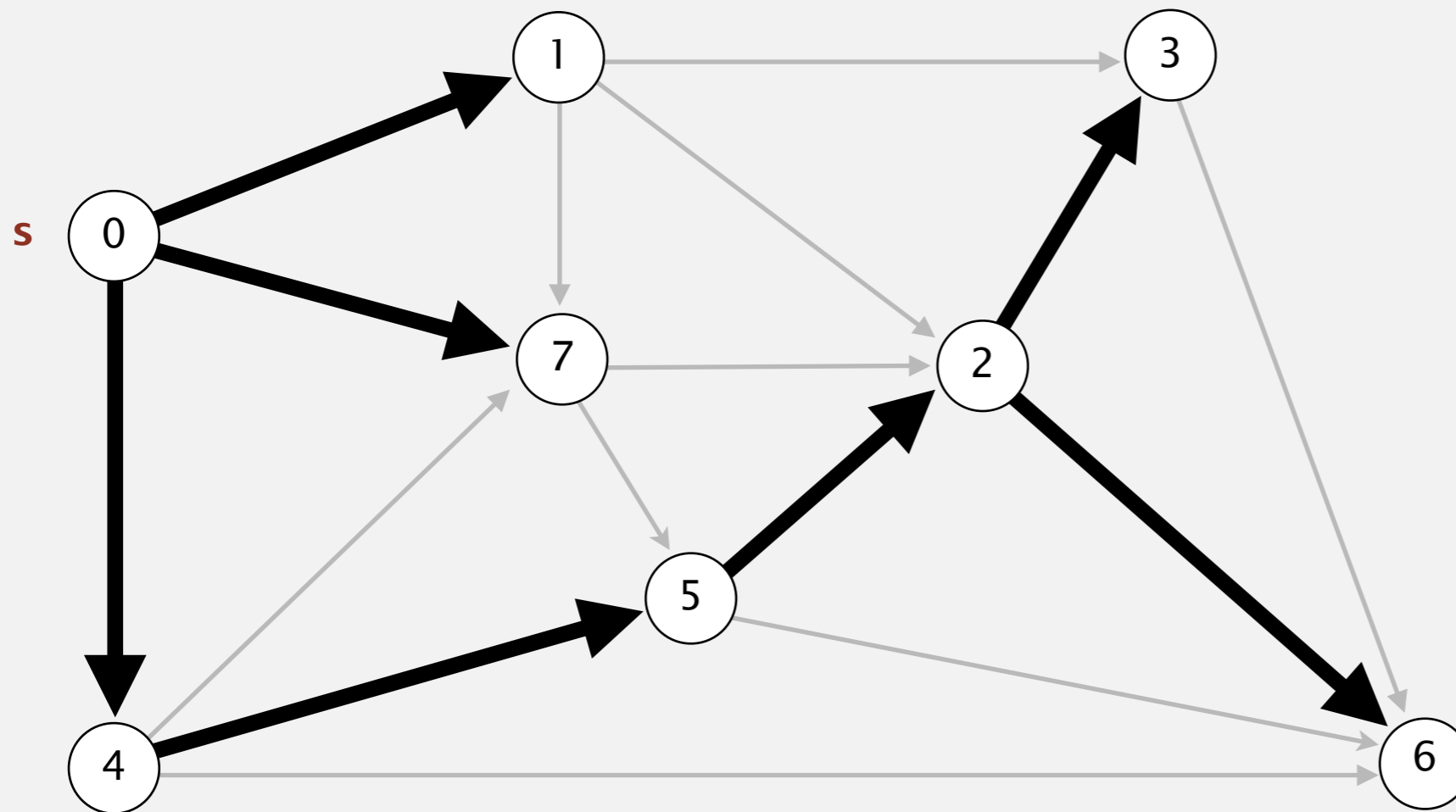


0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
6→7	6.0
7→2	7.0

an edge-weighted digraph

# Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from  $s$  (non-tree vertex with the lowest  $\text{distTo}[]$  value).
- Add vertex to tree and relax all edges adjacent from that vertex.

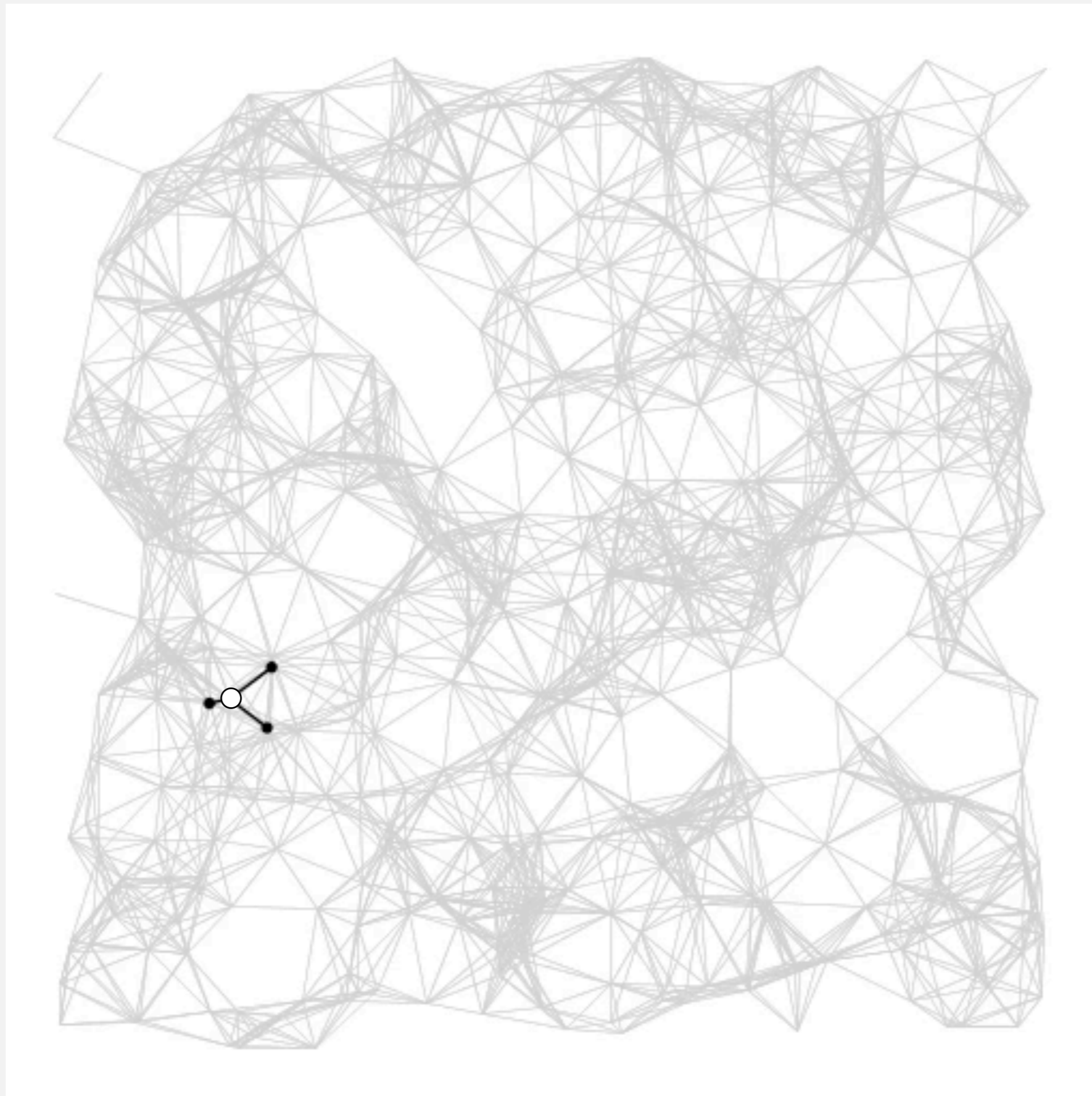


$v$	$\text{distTo}[]$	$\text{edgeTo}[]$
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

**shortest-paths tree from vertex  $s$**

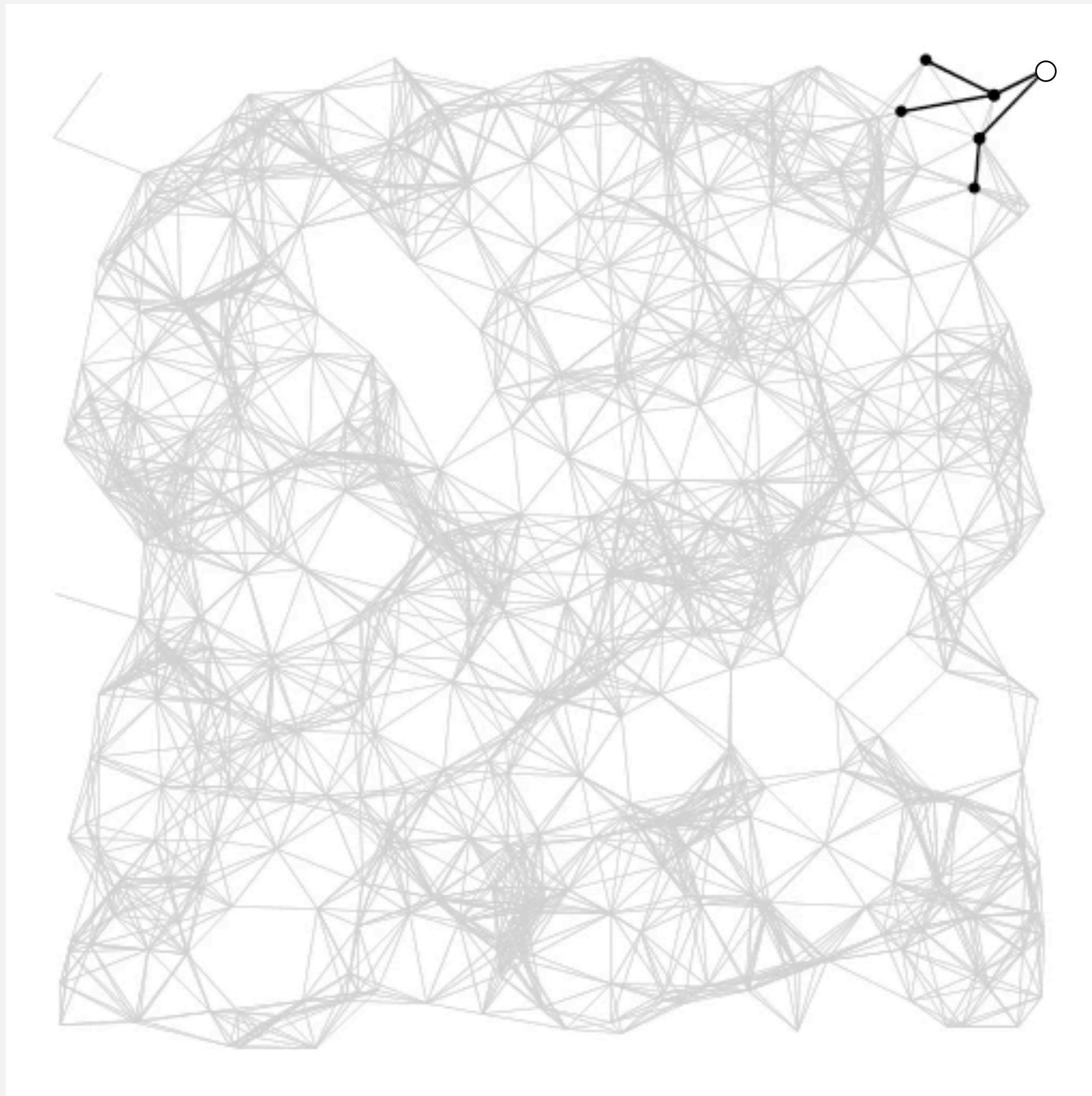
# Dijkstra's algorithm visualization

---



# Dijkstra's algorithm visualization

---



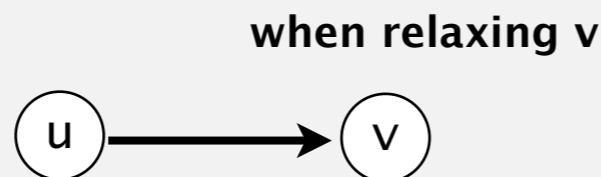
# Dijkstra's algorithm: correctness proof 1

---

**Proposition.** Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

**Pf.**

- Each edge  $e = v \rightarrow w$  is relaxed exactly once (when vertex  $v$  is relaxed), leaving  $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$ .
- Inequality holds until algorithm terminates because:
  - $\text{distTo}[w]$  cannot increase  $\leftarrow \text{distTo}[]$  values are monotone decreasing
  - $\text{distTo}[v]$  will not change  $\leftarrow$  we choose lowest  $\text{distTo}[]$  value at each step (and edge weights are nonnegative)



if  $u$  has not yet been relaxed,  
then  $\text{distTo}[u] \geq \text{distTo}[v]$

- Thus, upon termination, shortest-paths optimality conditions hold. ■

# Dijkstra's algorithm: Java implementation

---

```
public class DijkstraSP
{
    private DirectedEdge[] edgeTo;
    private double[] distTo;
    private IndexMinPQ<Double> pq;

    public DijkstraSP(EdgeWeightedDigraph G, int s)
    {
        edgeTo = new DirectedEdge[G.V()];
        distTo = new double[G.V()];
        pq = new IndexMinPQ<Double>(G.V());

        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;

        pq.insert(s, 0.0);
        while (!pq.isEmpty())
        {
            int v = pq.delMin();
            for (DirectedEdge e : G.adj(v))
                relax(e);
        }
    }
}
```

← relax vertices in order  
of distance from s



# Dijkstra's algorithm: Java implementation

---

```
private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
        if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
        else pq.insert(w, distTo[w]);
    }
}
```

← update PQ

# Dijkstra's algorithm: which priority queue?

---

Depends on PQ implementation:  $V$  insert,  $V$  delete-min,  $E$  decrease-key.

PQ implementation	insert	delete-min	decrease-key	total
<b>unordered array</b>	1	$V$	1	$V^2$
<b>binary heap</b>	$\log V$	$\log V$	$\log V$	$E \log V$
<b>d-way heap</b>	$\log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
<b>Fibonacci heap</b>	$1^\dagger$	$\log V^\dagger$	$1^\dagger$	$E + V \log V$

$\dagger$  amortized

## Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

# Computing a spanning tree in a graph

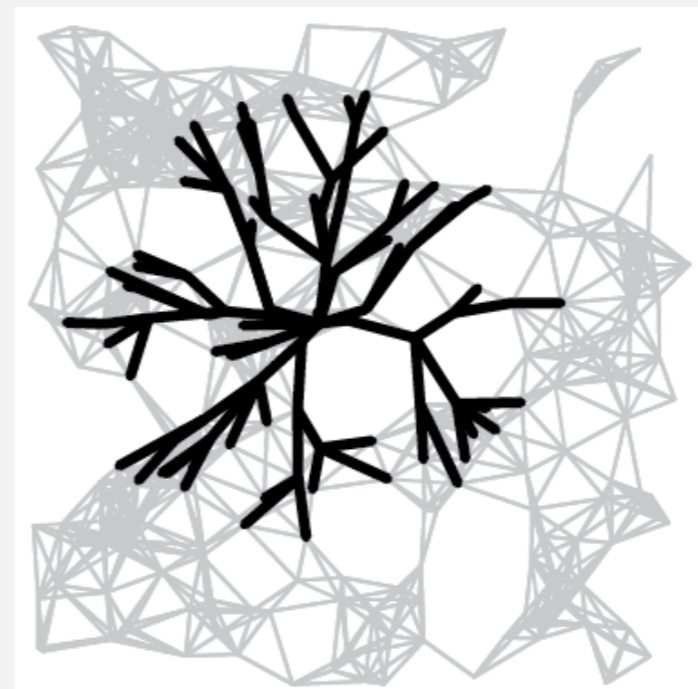
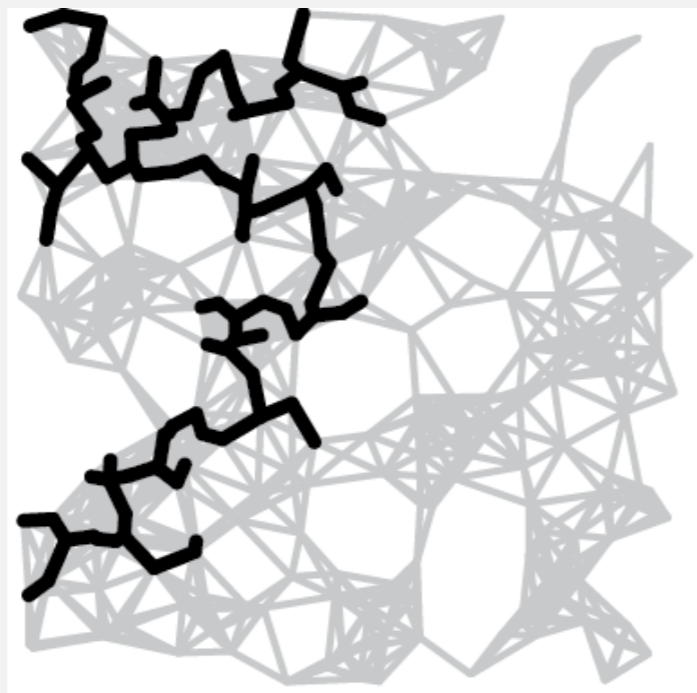
---

## Dijkstra's algorithm seem familiar?

- Prim's algorithm is essentially the same algorithm.
- Both are in a family of algorithms that compute a spanning tree.

**Main distinction:** rule used to choose next vertex for the tree.

- Prim: Closest vertex to the **tree** (via an undirected edge).
- Dijkstra: Closest vertex to the **source** (via a directed path).



**Note:** DFS and BFS are also in this family of algorithms.



# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 4.4 SHORTEST PATHS

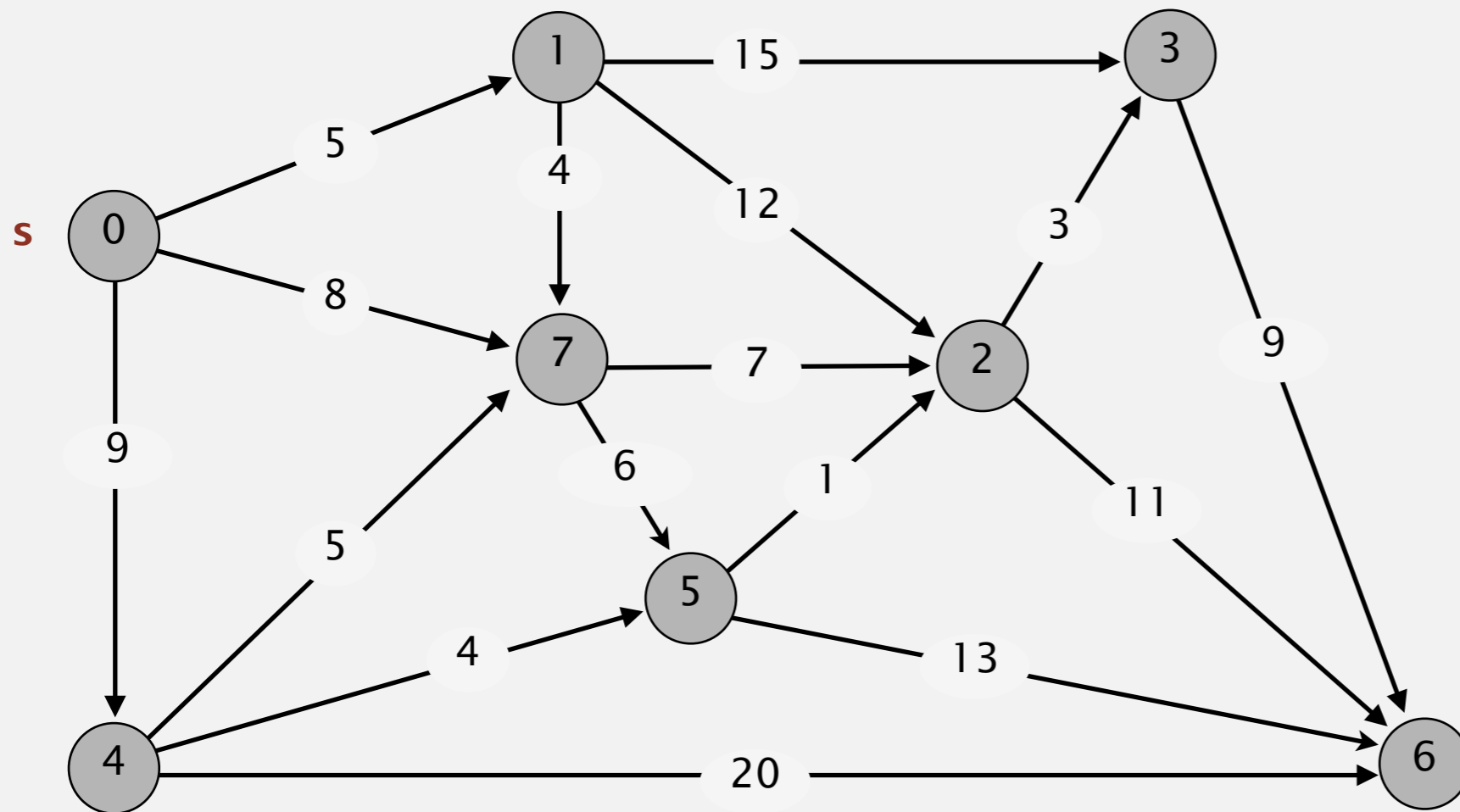
---

- ▶ *APIs*
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ *edge-weighted DAGs*
- ▶ *negative weights*

# Acyclic edge-weighted digraphs

---

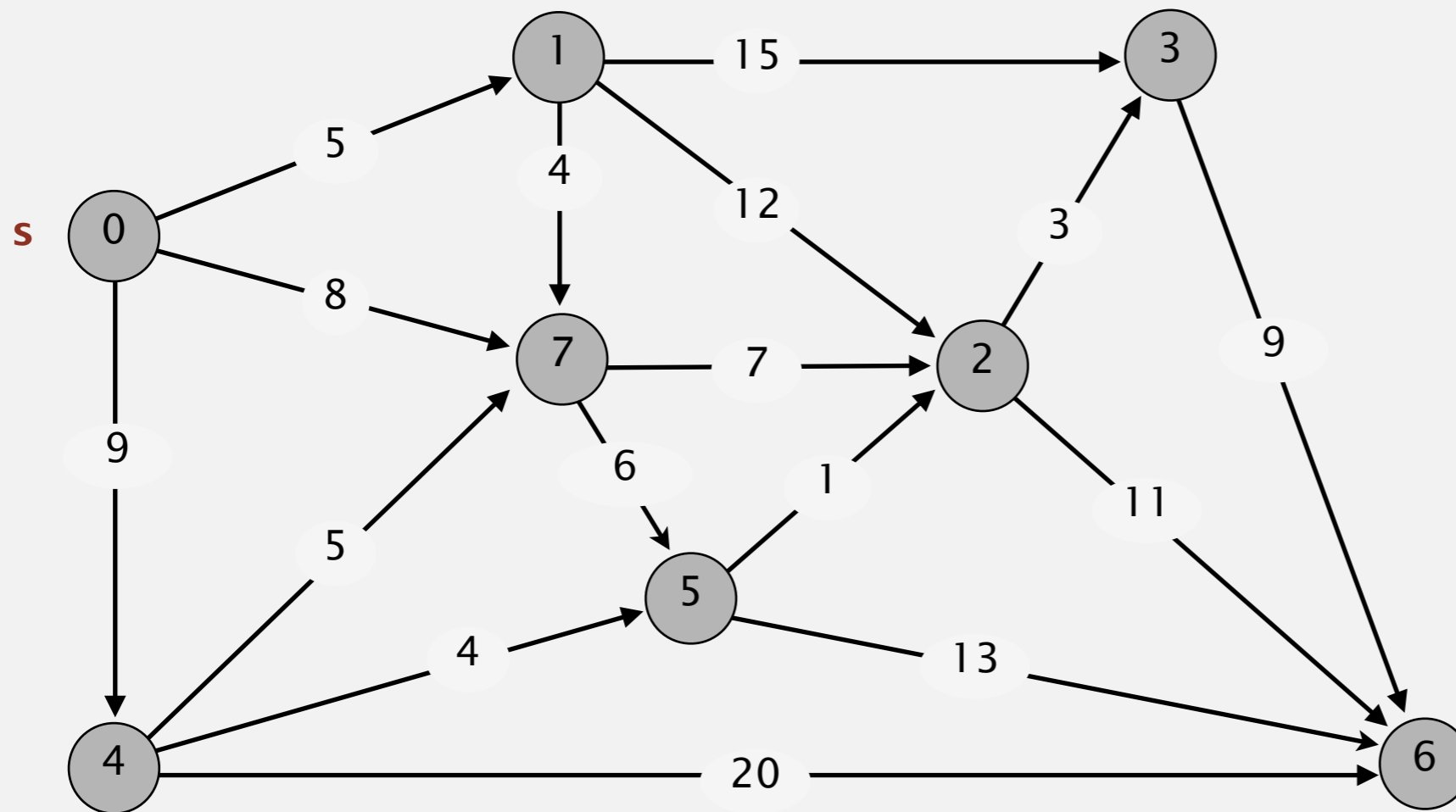
Q. Suppose that an edge-weighted digraph has no directed cycles. Is it easier to find shortest paths than in a general digraph?



A. Yes!

# Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges adjacent from that vertex.



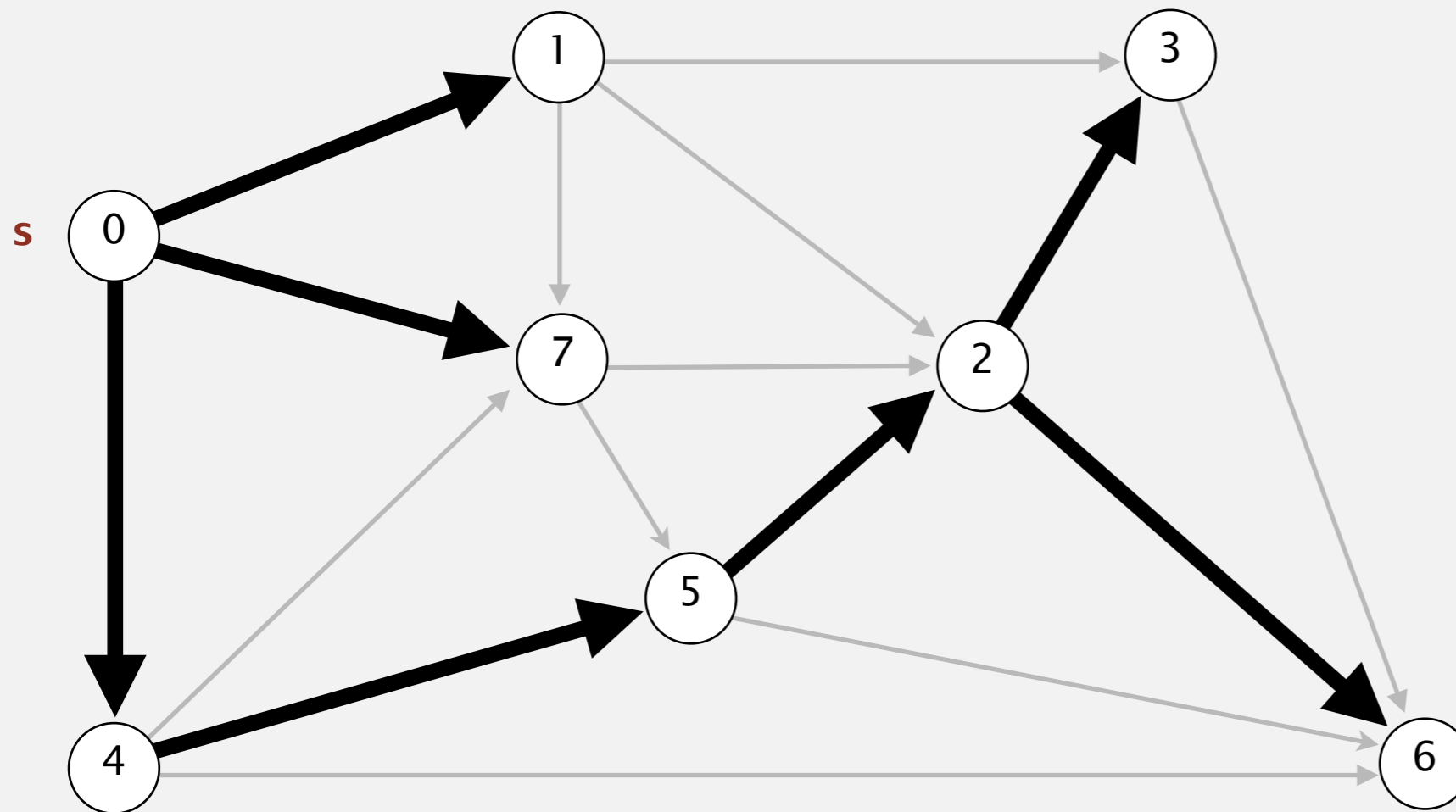
0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

an edge-weighted DAG



# Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges adjacent from that vertex.



	0	1	4	7	5	2	3	6
v	distTo[]	edgeTo[]						
0	0.0	-						
1	5.0	0→1						
2	14.0	5→2						
3	17.0	2→3						
4	9.0	0→4						
5	13.0	4→5						
6	25.0	2→6						
7	8.0	0→7						

shortest-paths tree from vertex s

# Shortest paths in edge-weighted DAGs

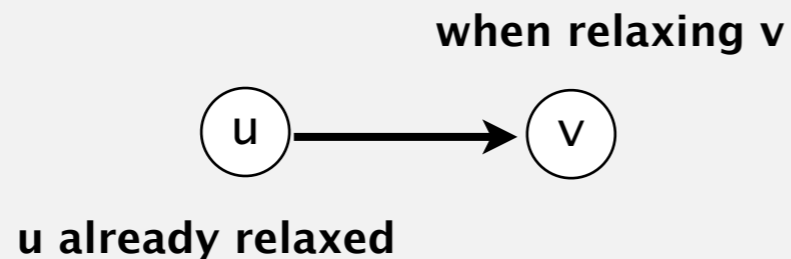
---

**Proposition.** Topological sort algorithm computes SPT in any edge-weighted DAG in time proportional to  $E + V$ .

edge weights  
can be negative!

**Pf.**

- Each edge  $e = v \rightarrow w$  is relaxed exactly once (when vertex  $v$  is relaxed), leaving  $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$ .
- Inequality holds until algorithm terminates because:
  - $\text{distTo}[w]$  cannot increase ←  $\text{distTo}[]$  values are monotone decreasing
  - $\text{distTo}[v]$  will not change ← because of topological order, no vertex adjacent to  $v$  will be relaxed after  $v$  is relaxed



- Thus, upon termination, shortest-paths optimality conditions hold. ■

# Shortest paths in edge-weighted DAGs

---

```
public class AcyclicSP
{
    private DirectedEdge[] edgeTo;
    private double[] distTo;

    public AcyclicSP(EdgeWeightedDigraph G, int s)
    {
        edgeTo = new DirectedEdge[G.V()];
        distTo = new double[G.V()];

        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;

        Topological topological = new Topological(G);
        for (int v : topological.order())
            for (DirectedEdge e : G.adj(v))
                relax(e);
    }
}
```

← topological order

# Content-aware resizing

---

**Seam carving.** [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.



Shai Avidan  
Mitsubishi Electric Research Lab  
Ariel Shamir  
The interdisciplinary Center & MERL

<http://www.youtube.com/watch?v=vIFCV2spKtg>

# Content-aware resizing

---

**Seam carving.** [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.



**In the wild.** Photoshop CS 5, Imagemagick, GIMP, ...

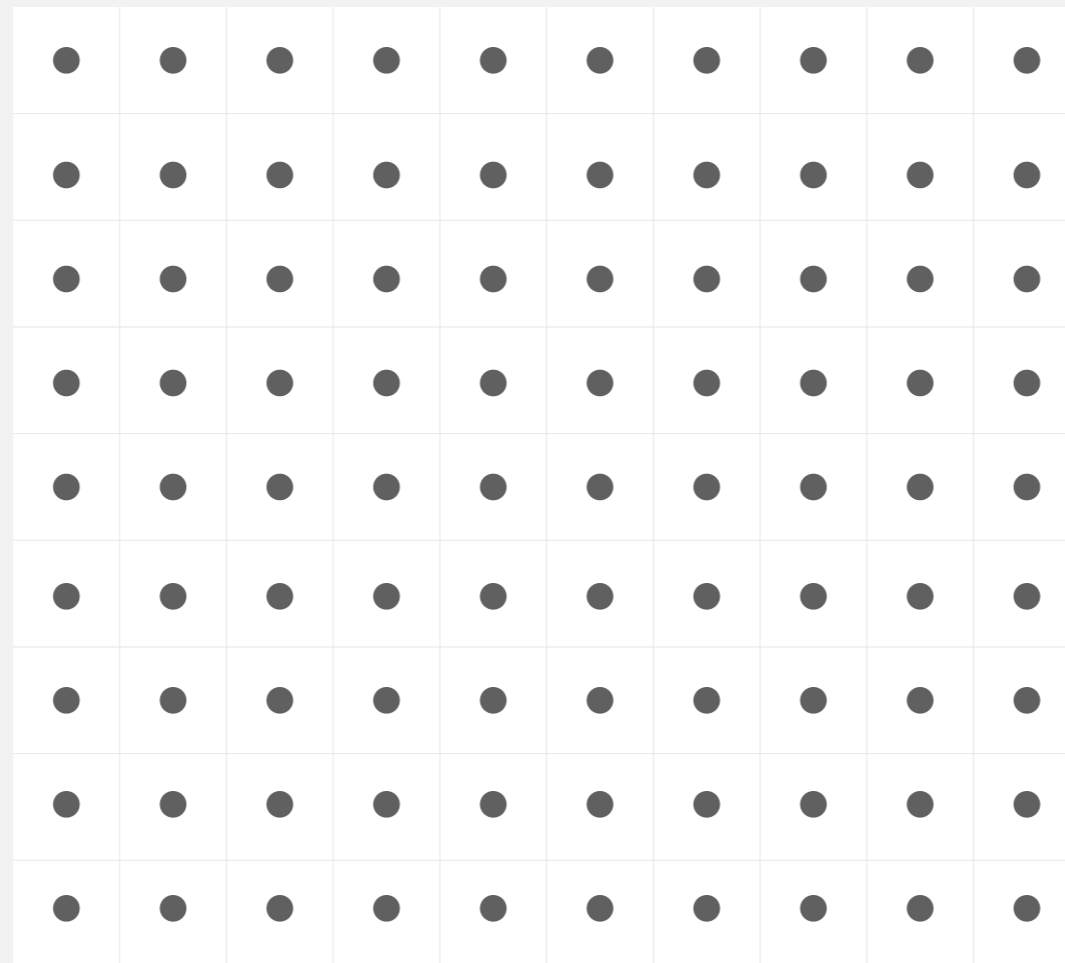


# Content-aware resizing

---

## To find vertical seam:

- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path (sum of vertex weights) from top to bottom.



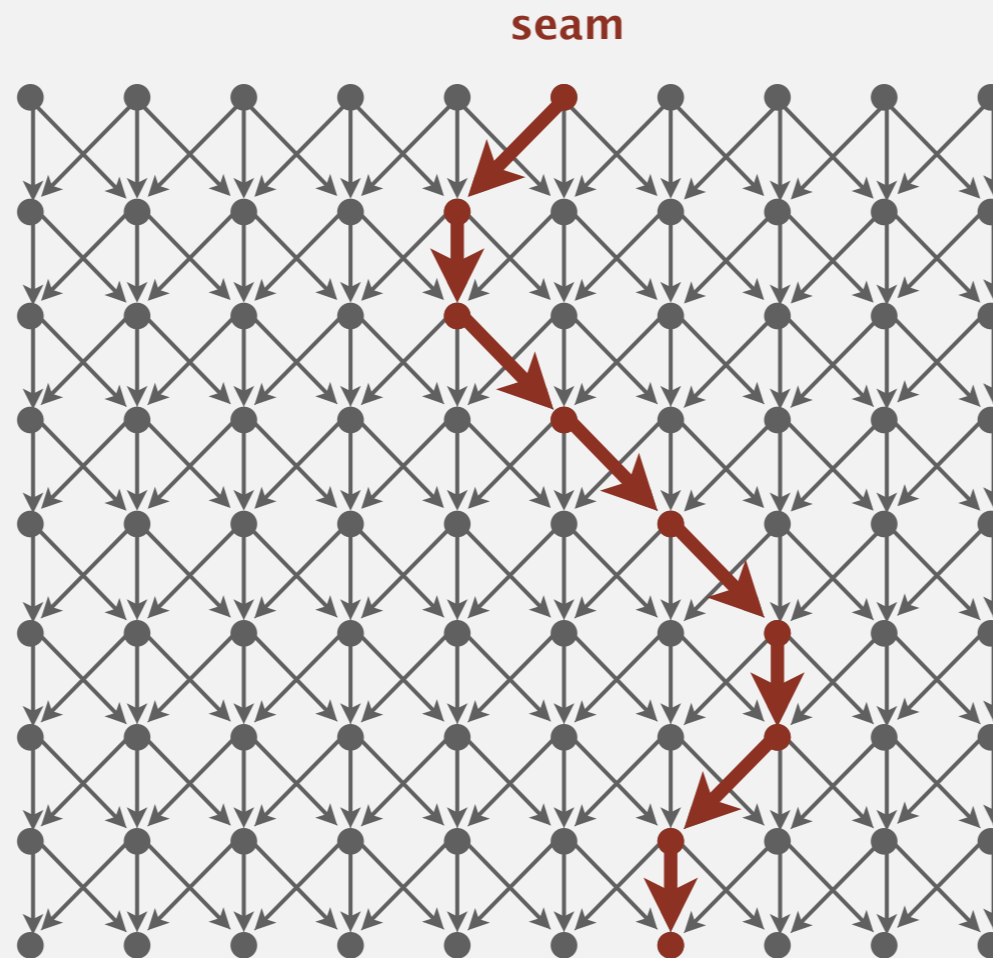


# Content-aware resizing

---

## To find vertical seam:

- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path (sum of vertex weights) from top to bottom.

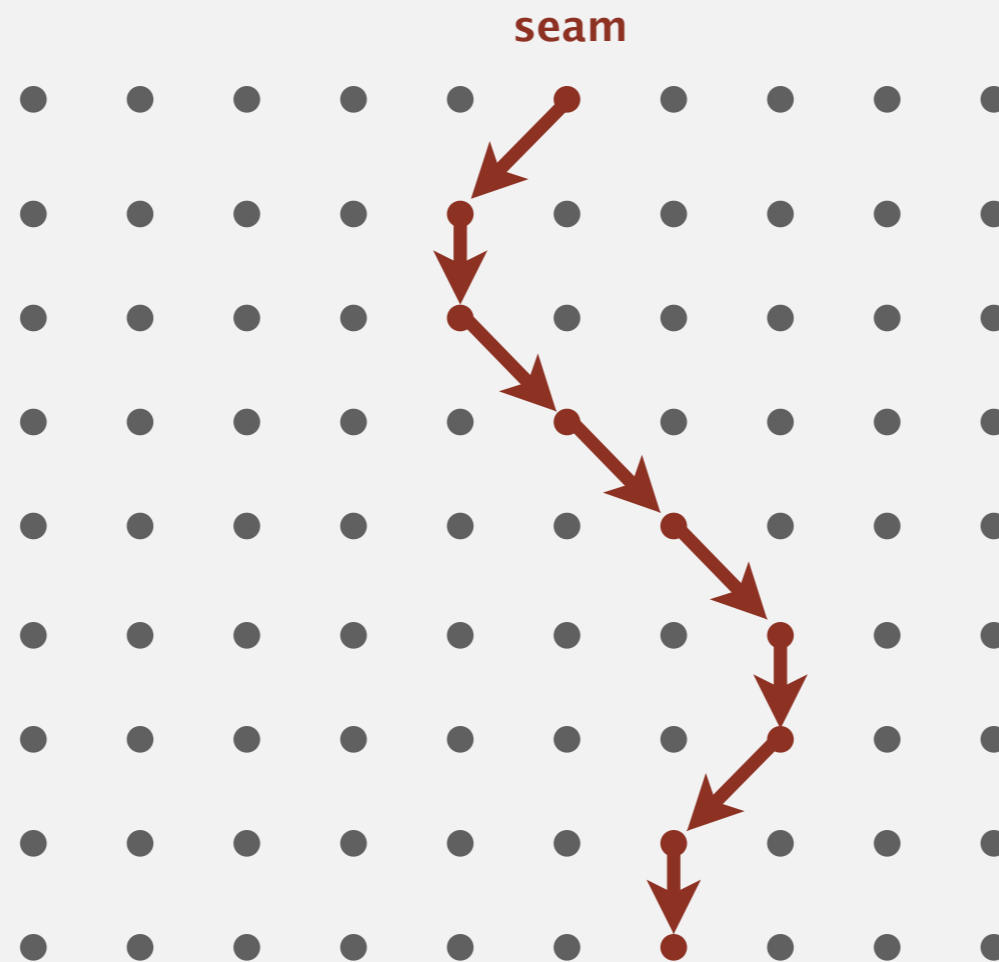


# Content-aware resizing

---

To remove vertical seam:

- Delete pixels on seam (one in each row).

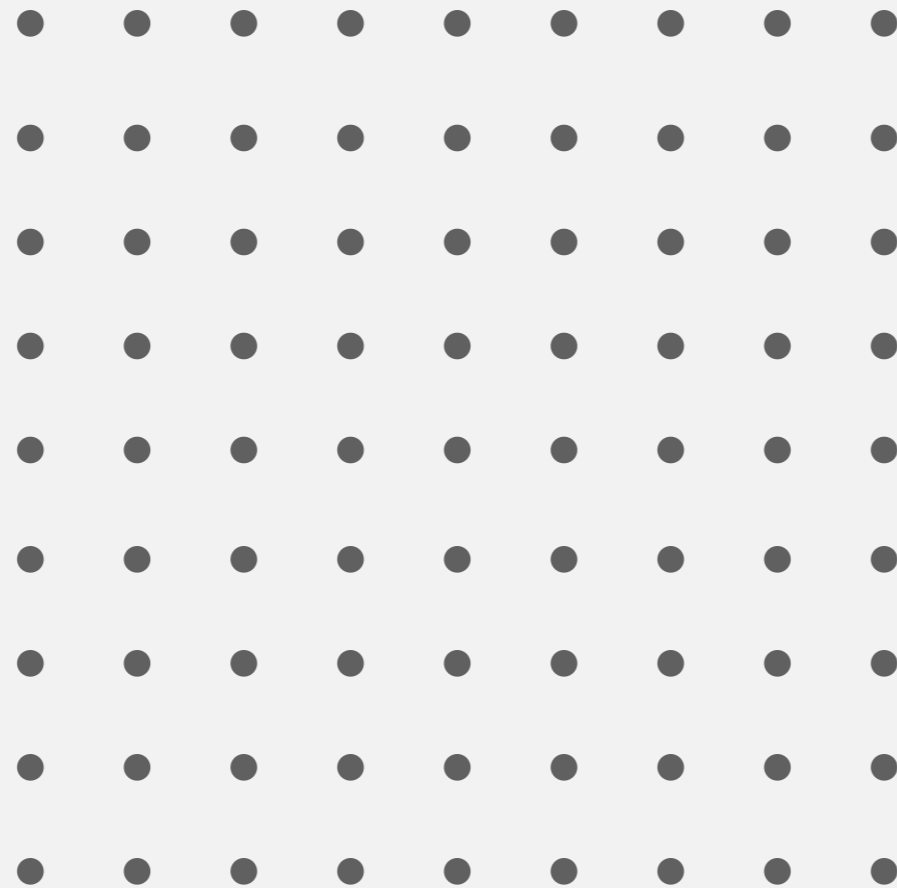


# Content-aware resizing

---

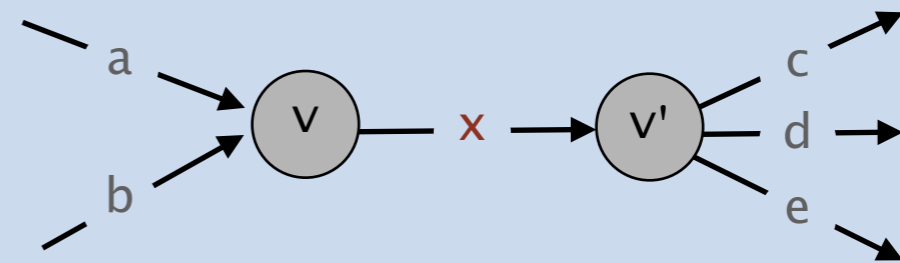
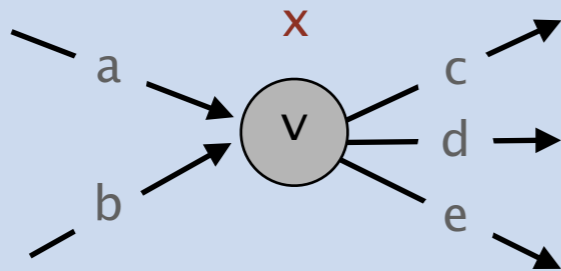
To remove vertical seam:

- Delete pixels on seam (one in each row).

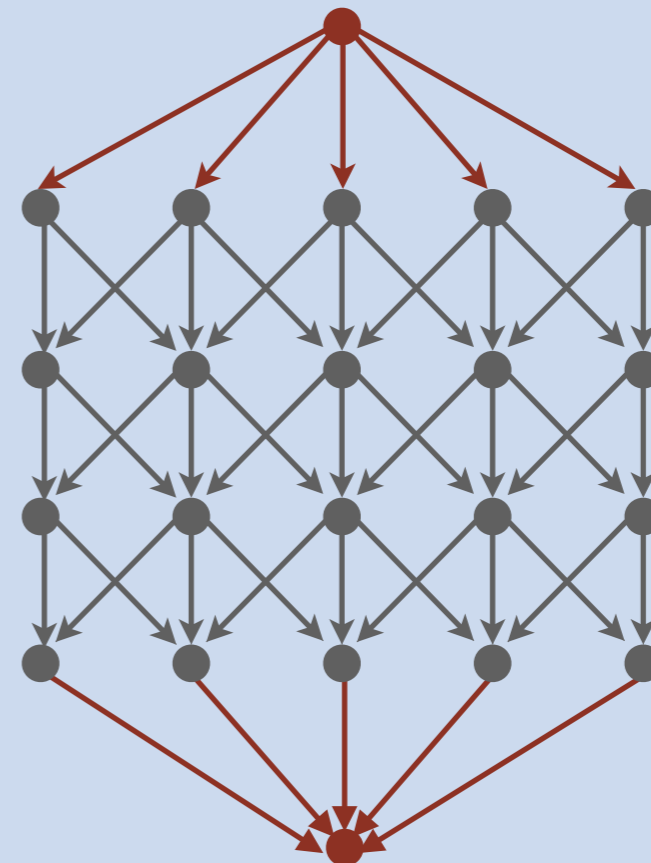
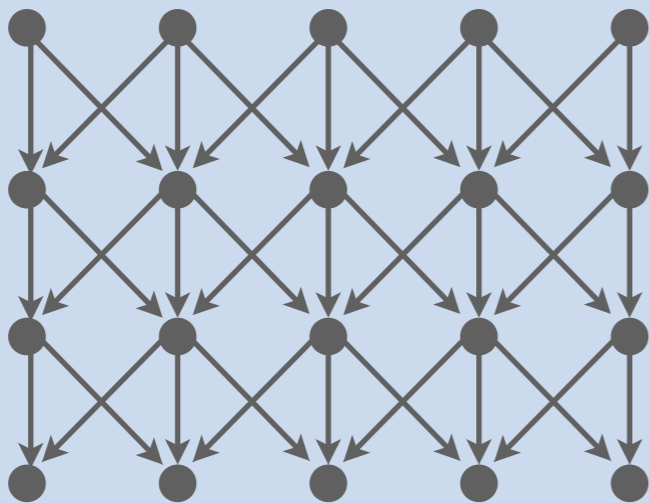


# SHORTEST PATH VARIANTS

Q1. How to model both vertex and edge weights?



Q2. How to model multiple sources and sinks?



# Longest paths in edge-weighted DAGs

Formulate as a shortest paths problem in edge-weighted DAGs.

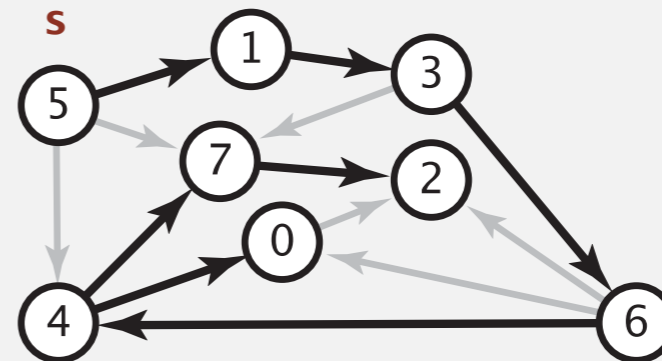
- Negate all weights.
  - Find shortest paths.
  - Negate weights in result.
- equivalent: reverse sense of equality in relax()

longest paths input

5->4	0.35
4->7	0.37
5->7	0.28
5->1	0.32
4->0	0.38
0->2	0.26
3->7	0.39
1->3	0.29
7->2	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93

shortest paths input

5->4	-0.35
4->7	-0.37
5->7	-0.28
5->1	-0.32
4->0	-0.38
0->2	-0.26
3->7	-0.39
1->3	-0.29
7->2	-0.34
6->2	-0.40
3->6	-0.52
6->0	-0.58
6->4	-0.93



**Key point.** Topological sort algorithm works even with negative weights.

# Longest paths in edge-weighted DAGs: application

**Parallel job scheduling.** Given a set of jobs with durations and precedence constraints, schedule the jobs (by finding a start time for each) so as to achieve the minimum completion time, while respecting the constraints.

<i>job</i>	<i>duration</i>	<i>must complete before</i>		
0	41.0	1	7	9
1	51.0	2		
2	50.0			
3	36.0			
4	38.0			
5	45.0			
6	21.0	3	8	
7	32.0	3	8	
8	32.0	2		
9	29.0	4	6	



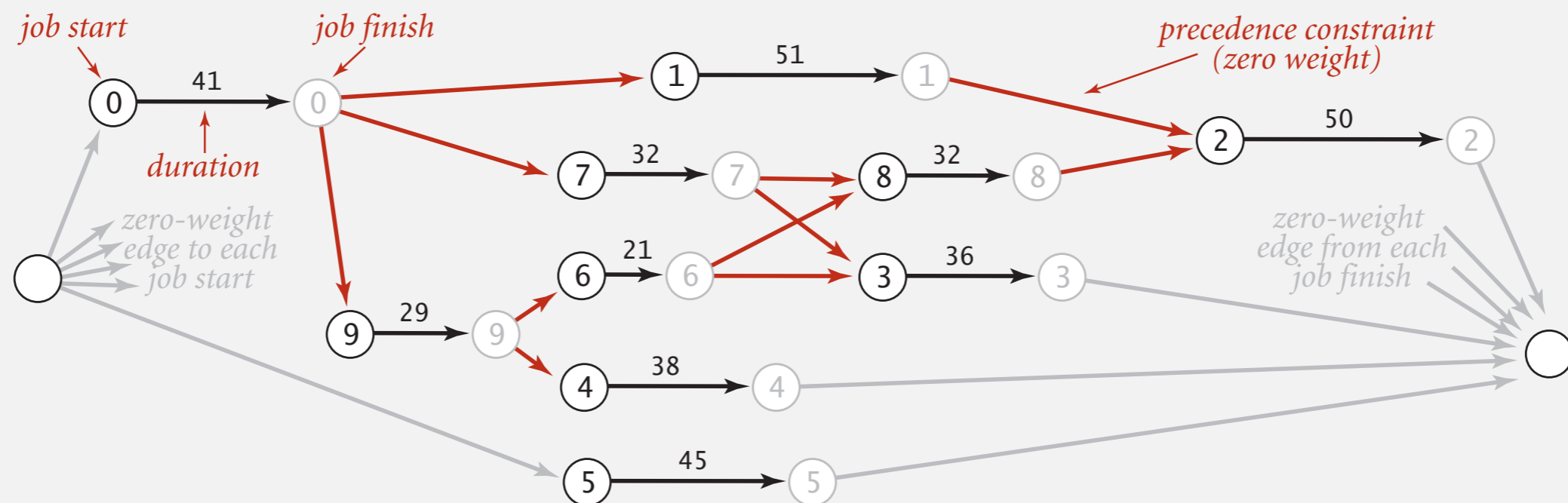
Parallel job scheduling solution

# Critical path method

**CPM.** To solve a parallel job-scheduling problem, create edge-weighted DAG:

- Source and sink vertices.
- Two vertices (begin and end) for each job.
- Three edges for each job.
  - begin to end (weighted by duration)
  - source to begin (0 weight)
  - end to sink (0 weight)
- One edge for each precedence constraint (0 weight).

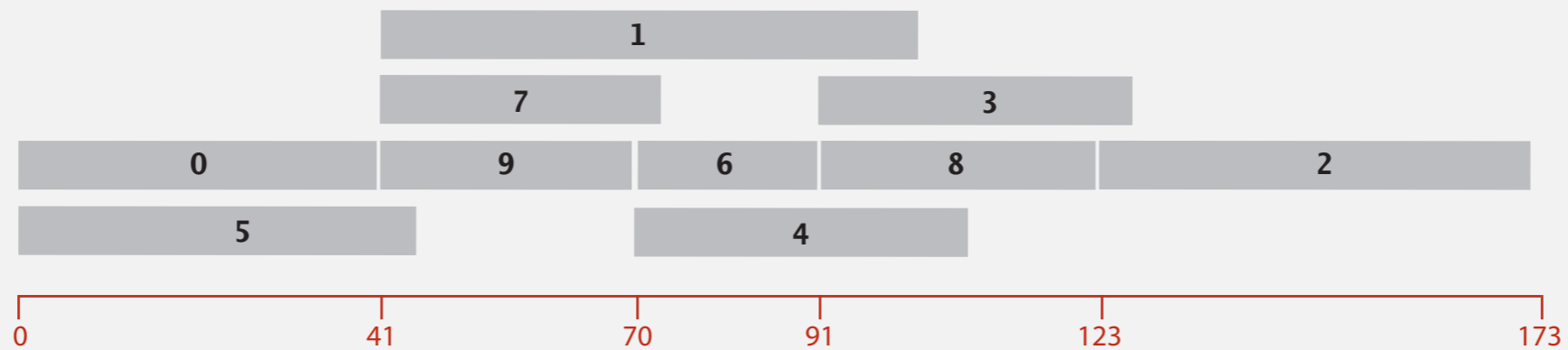
<i>job</i>	<i>duration</i>	<i>must complete before</i>		
0	41.0	1	7	9
1	51.0	2		
2	50.0			
3	36.0			
4	38.0			
5	45.0			
6	21.0	3	8	
7	32.0	3	8	
8	32.0	2		
9	29.0	4	6	



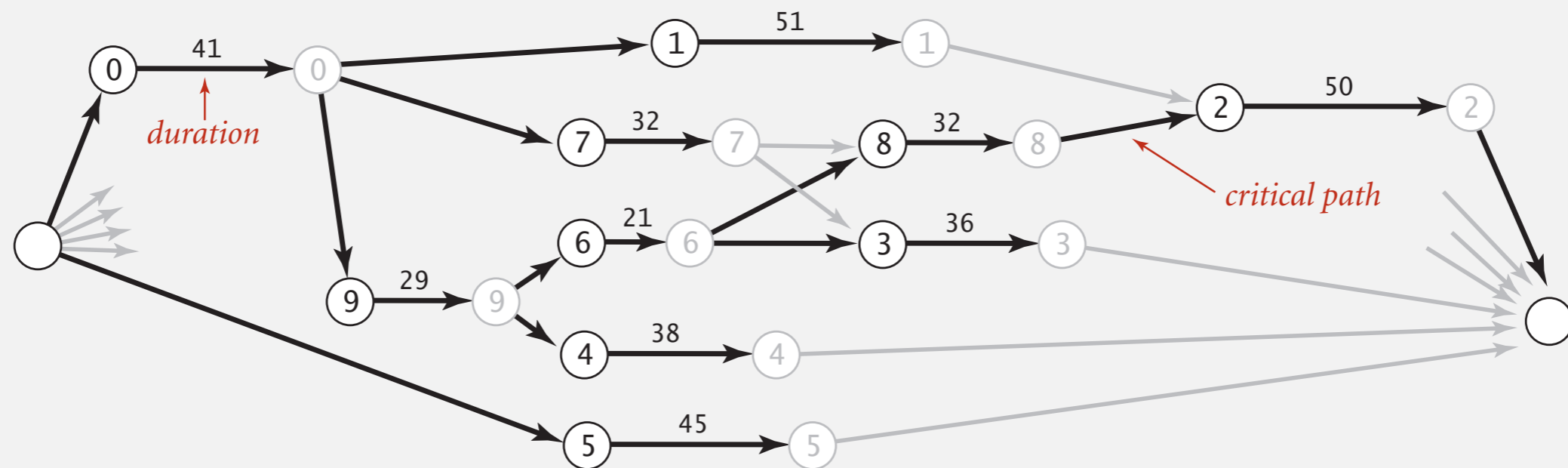


# Critical path method

CPM. Use **longest path** from the source to schedule each job.



Parallel job scheduling solution





# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

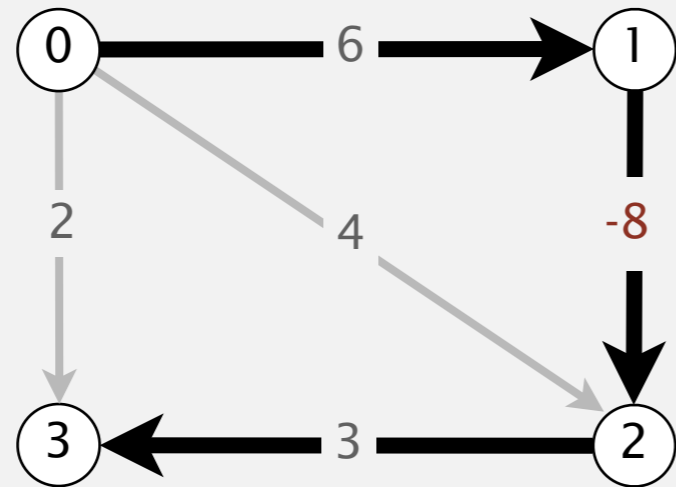
## 4.4 SHORTEST PATHS

---

- ▶ *APIs*
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ *edge-weighted DAGs*
- ▶ *negative weights*

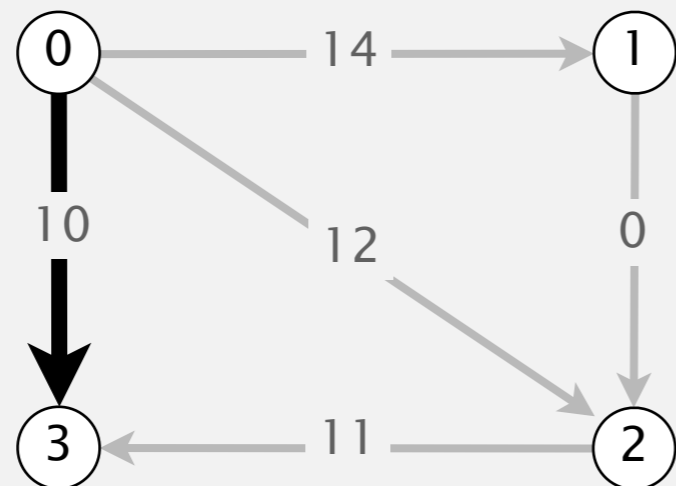
# Shortest paths with negative weights: failed attempts

**Dijkstra.** Doesn't work with negative edge weights.



Dijkstra selects the vertices in the order 0, 3, 2, 1  
But shortest path from 0 to 3 is 0→1→2→3.

**Re-weighting.** Add a constant to every edge weight doesn't work.



Adding 8 to each edge weight changes the  
shortest path from 0→1→2→3 to 0→3.

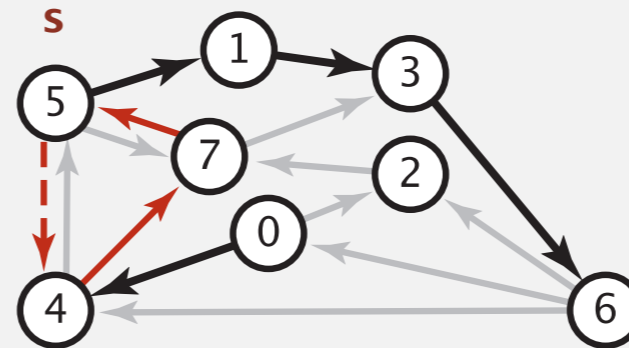
**Conclusion.** Need a different algorithm.

# Negative cycles

A **negative cycle** is a directed cycle whose sum of edge weights is negative.

**digraph**

4→5	0.35
5→4	-0.66
4→7	0.37
5→7	0.28
7→5	0.28
5→1	0.32
0→4	0.38
0→2	0.26
7→3	0.39
1→3	0.29
2→7	0.34
6→2	0.40
3→6	0.52
6→0	0.58
6→4	0.93



**negative cycle**  $(-0.66 + 0.37 + 0.28)$

5→4→7→5

**shortest path from 0 to 6**

0→4→7→5→4→7→5...→1→3→6

**Proposition.** A SPT exists iff no negative cycles (reachable from  $s$ ).

# Bellman-Ford algorithm

---

## Bellman-Ford algorithm

---

Initialize  $\text{distTo}[s] = 0$  and  $\text{distTo}[v] = \infty$  for all other vertices.

Repeat  $V$  times:

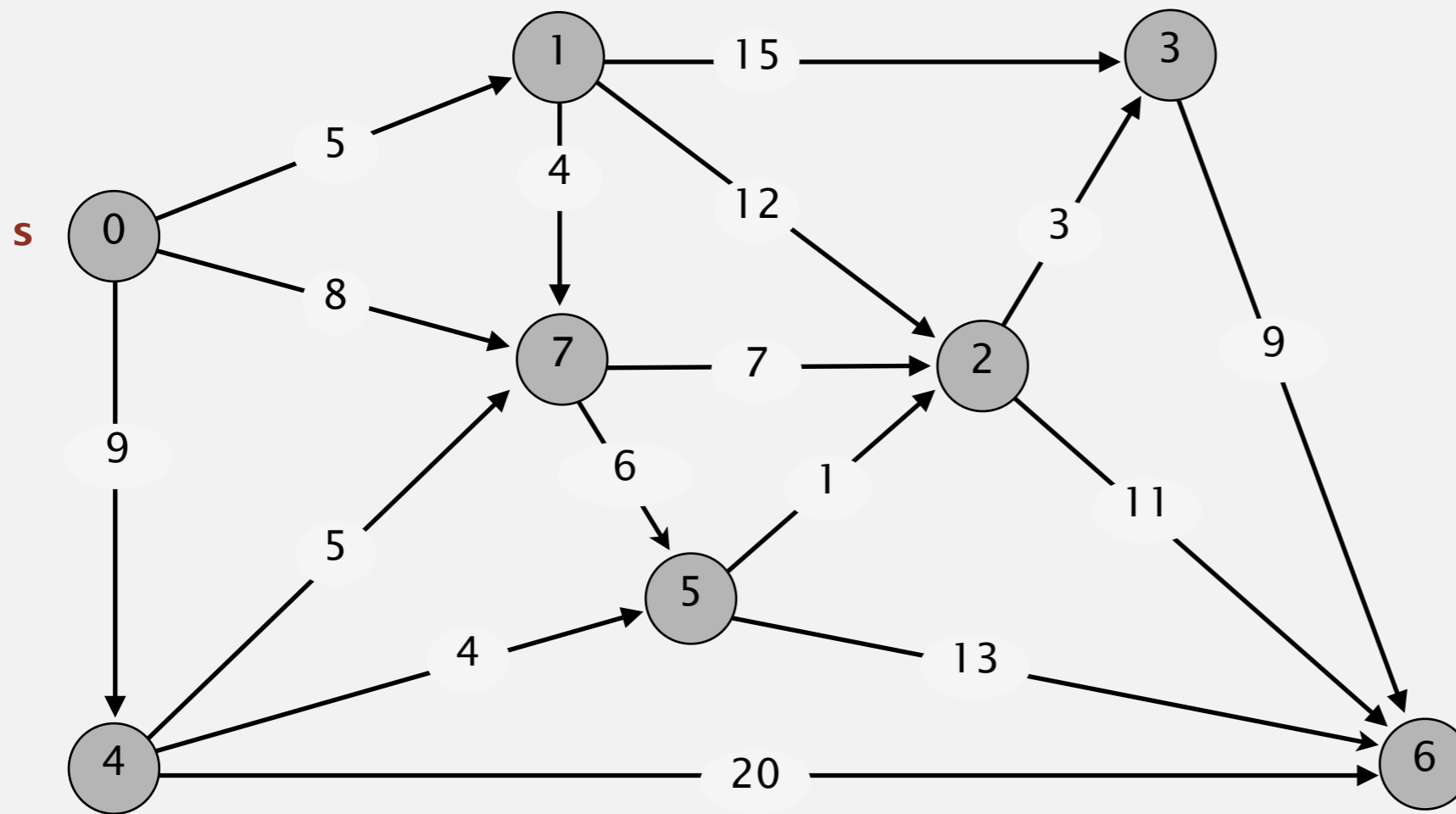
- Relax each edge.
- 

```
for (int i = 0; i < G.V(); i++)  
    for (int v = 0; v < G.V(); v++)  
        for (DirectedEdge e : G.adj(v))  
            relax(e);
```

← pass i (relax each edge)

# Bellman-Ford algorithm demo

Repeat  $V$  times: relax all  $E$  edges.

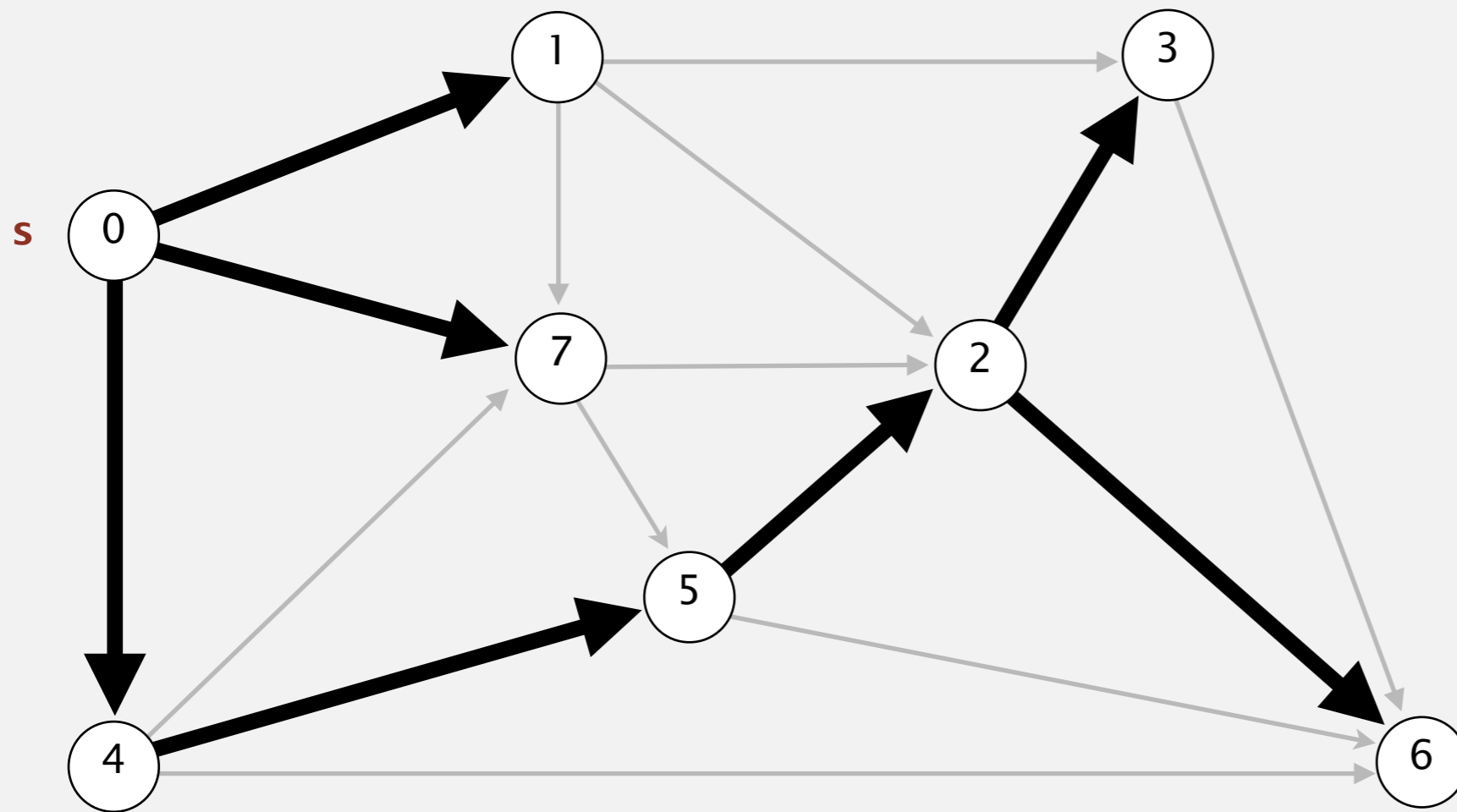


0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

an edge-weighted digraph

# Bellman-Ford algorithm demo

Repeat  $V$  times: relax all  $E$  edges.



$v$	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

**shortest-paths tree from vertex s**

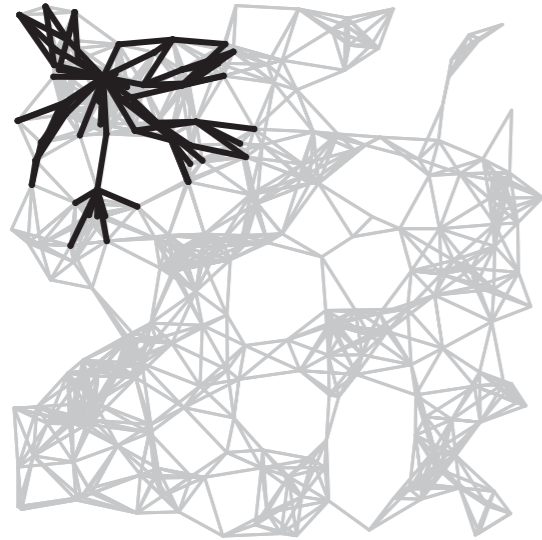


# Bellman-Ford algorithm: visualization

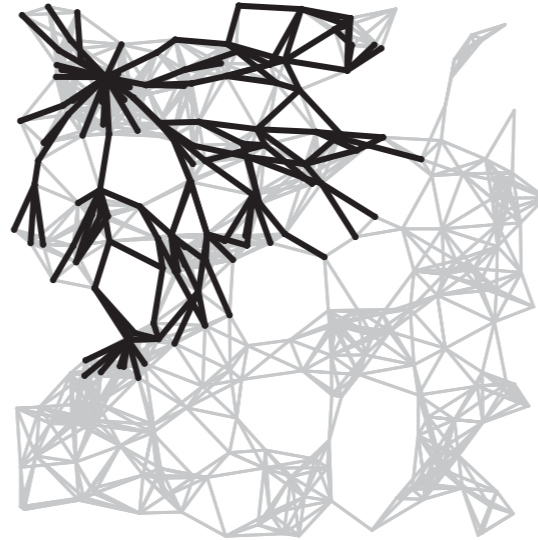
---

passes

4



7



10



13



SPT



# Bellman-Ford algorithm: analysis

---

## Bellman-Ford algorithm

---

Initialize  $\text{distTo}[s] = 0$  and  $\text{distTo}[v] = \infty$  for all other vertices.

Repeat  $V$  times:

- Relax each edge.
- 

**Proposition.** Dynamic programming algorithm computes SPT in any edge-weighted digraph with no negative cycles in time proportional to  $E \times V$ .

**Pf idea.** After pass  $i$ , found shortest path to each vertex  $v$  for which the shortest path from  $s$  to  $v$  contains  $i$  edges (or fewer).

# Bellman-Ford algorithm: practical improvement

---

**Observation.** If  $\text{distTo}[v]$  does not change during pass  $i$ , no need to relax any edge adjacent from  $v$  in pass  $i+1$ .

**FIFO implementation.** Maintain **queue** of vertices whose  $\text{distTo}[]$  changed.



be careful to keep at most one copy  
of each vertex on queue (why?)

**Overall effect.**

- The running time is still proportional to  $E \times V$  in worst case.
- But much faster than that in practice.

# Single source shortest-paths implementation: cost summary

---

algorithm	restriction	typical case	worst case	extra space
<b>topological sort</b>	no directed cycles	$E + V$	$E + V$	$V$
<b>Dijkstra (binary heap)</b>	no negative weights	$E \log V$	$E \log V$	$V$
<b>Bellman-Ford</b>	no negative cycles	$E V$	$E V$	$V$
<b>Bellman-Ford (queue-based)</b>		$E + V$	$E V$	$V$

**Remark 1.** Directed cycles make the problem harder.

**Remark 2.** Negative weights make the problem harder.

**Remark 3.** Negative cycles makes the problem intractable.

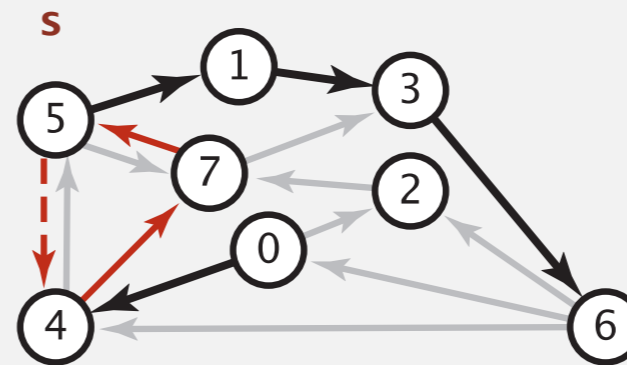
# Finding a negative cycle

Negative cycle. Add two methods to the API for SP.

boolean	hasNegativeCycle()	<i>is there a negative cycle?</i>
Iterable <DirectedEdge>	negativeCycle()	<i>negative cycle reachable from s</i>

digraph

- 4->5 0.35
- 5->4 -0.66
- 4->7 0.37
- 5->7 0.28
- 7->5 0.28
- 5->1 0.32
- 0->4 0.38
- 0->2 0.26
- 7->3 0.39
- 1->3 0.29
- 2->7 0.34
- 6->2 0.40
- 3->6 0.52
- 6->0 0.58
- 6->4 0.93

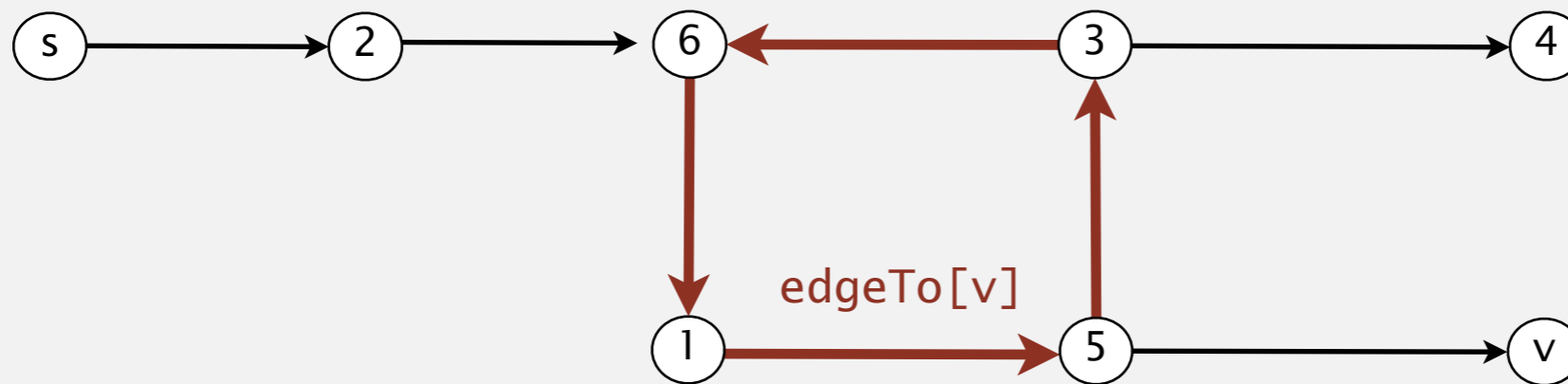


negative cycle (-0.66 + 0.37 + 0.28)  
5->4->7->5

# Finding a negative cycle

---

**Observation.** If there is a negative cycle, Bellman-Ford gets stuck in loop, updating `distTo[]` and `edgeTo[]` entries of vertices in the cycle.



**Proposition.** If Bellman-Ford updates any vertex  $v$  in pass  $v$ , there exists a negative cycle (and can trace `edgeTo[v]` entries back to find one).

**In practice.** Check for negative cycles more frequently.


# Negative cycle application: arbitrage detection

---

**Problem.** Given table of exchange rates, is there an arbitrage opportunity?

	USD	EUR	GBP	CHF	CAD
USD	1	0.741	0.657	1.061	1.011
EUR	1.350	1	0.888	1.433	1.366
GBP	1.521	1.126	1	1.614	1.538
CHF	0.943	0.698	0.620	1	0.953
CAD	0.995	0.732	0.650	1.049	1

**Ex.** \$1,000  $\Rightarrow$  741 Euros  $\Rightarrow$  1,012.206 Canadian dollars  $\Rightarrow$  \$1,007.14497.

$$1000 \times 0.741 \times 1.366 \times 0.995 = 1007.14497$$


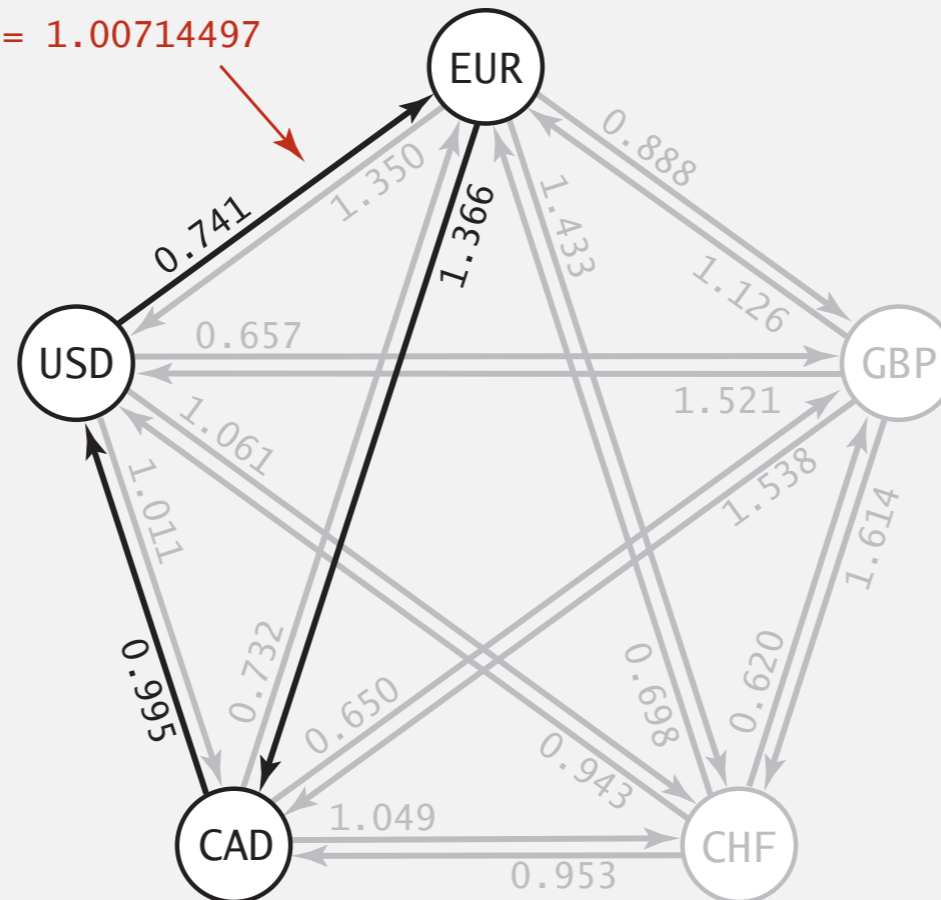


# Negative cycle application: arbitrage detection

## Currency exchange graph.

- Vertex = currency.
- Edge = transaction, with weight equal to exchange rate.
- Find a directed cycle whose product of edge weights is  $> 1$ .

$$0.741 * 1.366 * .995 = 1.00714497$$

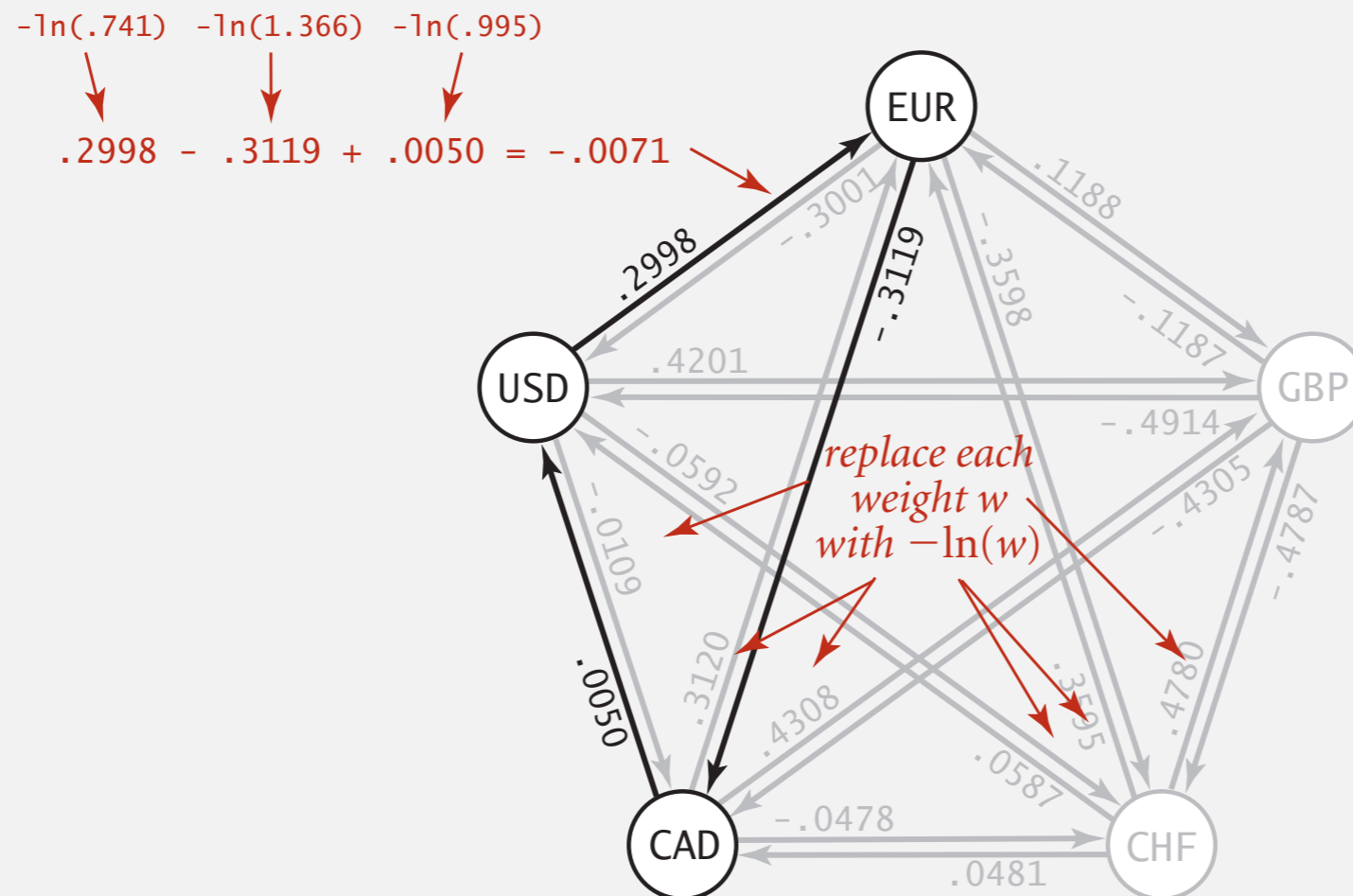


**Challenge.** Express as a negative cycle detection problem.

# Negative cycle application: arbitrage detection

Model as a negative cycle detection problem by taking logs.

- Let weight of edge  $v \rightarrow w$  be  $-\ln$  (exchange rate from currency  $v$  to  $w$ ).
- Multiplication turns to addition;  $> 1$  turns to  $< 0$ .
- Find a directed cycle whose sum of edge weights is  $< 0$  (negative cycle).



**Remark.** Fastest algorithm is extraordinarily valuable!

# Shortest paths summary

---

## Nonnegative weights.

- Arises in many applications.
- Dijkstra's algorithm is nearly linear-time.

## Acyclic edge-weighted digraphs.

- Arise in some applications.
- Topological sort algorithm is linear time.
- Edge weights can be negative.

## Negative weights and negative cycles.

- Arise in some applications.
- Bellman-Ford is quadratic in worst case.
- If no negative cycles, can find shortest paths via Bellman-Ford.
- If negative cycles, can find one via Bellman-Ford.

Shortest-paths is a broadly useful problem-solving model.