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2.4 PRIORITY QUEUES

- API and elementary implementations
- binary heaps
- heapsort
- event-driven simulation

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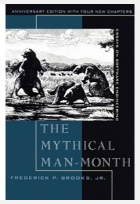
Collections

A collection is a data type that stores a group of items.

data type	key operations	data structure
stack	PUSH, POP	*linked list, resizing array*
queue	ENQUEUE, DEQUEUE	*linked list, resizing array*
priority queue	INSERT, DELETE-MAX	*binary heap*
symbol table	PUT, GET, DELETE	*binary search tree, hash table*
set	ADD, CONTAINS, DELETE	*binary search tree, hash table*

> *" Show me your code and conceal your data structures, and I shall continue to be mystified. Show me your data structures, and I won't usually need your code; it'll be obvious."* — Fred Brooks



Priority queue

Collections. Insert and delete items. Which item to delete?

Stack. Remove the item most recently added.
Queue. Remove the item least recently added.
Randomized queue. Remove a random item.

Priority queue. Remove the largest (or smallest) item.

operation	*argument*	*return value*
insert	P	
insert	Q	
insert	E	
remove max		Q
insert	X	
insert	A	
insert	M	
remove max		X
insert	P	
insert	L	
insert	E	
remove max		P

Priority queue API

Requirement. Items are generic; they must also be `Comparable`.

Key must be Comparable
(bounded type parameter)

`public class MaxPQ<Key extends Comparable<Key>>`

	`MaxPQ()`	*create an empty priority queue*
	`MaxPQ(Key[] a)`	*create a priority queue with given keys*
`void`	`insert(Key v)`	*insert a key into the priority queue*
`Key`	`delMax()`	*return and remove the largest key*
`boolean`	`isEmpty()`	*is the priority queue empty?*
`Key`	`max()`	*return the largest key*
`int`	`size()`	*number of entries in the priority queue*

Priority queue: applications

Event-driven simulation.	[customers in a line, colliding particles]
Numerical computation.	[reducing roundoff error]
Discrete optimization.	[bin packing, scheduling]
Artificial intelligence.	[A* search]
Computer networks.	[web cache]
Operating systems.	[load balancing, interrupt handling]
Data compression.	[Huffman codes]
Graph searching.	[Dijkstra's algorithm, Prim's algorithm]
Number theory.	[sum of powers]
Spam filtering.	[Bayesian spam filter]
Statistics.	[online median in data stream]

Generalizes: stack, queue, randomized queue.

Priority queue: client example

Challenge. Find the largest M items in a stream of N items.

- Fraud detection: isolate $$ transactions.
- NSA monitoring: flag most suspicious documents.

N huge, M large

Constraint. Not enough memory to store N items.

```
% more transactions.txt
Turing      6/17/1990   644.08
vonNeumann  3/26/2002  4121.85
Dijkstra    8/22/2007  2678.40
vonNeumann  1/11/1999  4409.74
Dijkstra   11/18/1995   837.42
Hoare       5/10/1993  3229.27
vonNeumann  2/12/1994  4732.35
Hoare       8/18/1992  4381.21
Turing      1/11/2002    66.10
Thompson    2/27/2000  4747.08
Turing      2/11/1991  2156.86
Hoare       8/12/2003  1025.70
vonNeumann 10/13/1993  2520.97
Dijkstra    9/10/2000   708.95
Turing     10/12/1993  3532.36
Hoare       2/10/2005  4050.20
```

```
% java TopM 5 < transactions.txt
Thompson    2/27/2000  4747.08
vonNeumann  2/12/1994  4732.35
vonNeumann  1/11/1999  4409.74
Hoare       8/18/1992  4381.21
vonNeumann  3/26/2002  4121.85
```

sort key

Priority queue: client example

Challenge. Find the largest M items in a stream of N items.

- Fraud detection: isolate $$ transactions.
- NSA monitoring: flag most suspicious documents.

Constraint. Not enough memory to store N items.

use a min-oriented pq

```
MinPQ<Transaction> pq = new MinPQ<Transaction>();

while (StdIn.hasNextLine())
{
   String line = StdIn.readLine();
   Transaction item = new Transaction(line);
   pq.insert(item);
   if (pq.size() > M)
      pq.delMin();
}
```

Transaction data
type is Comparable
(ordered by $$)

pq now contains
largest M items

Priority queue: client example

Challenge. Find the largest M items in a stream of N items.

implementation	time	space
sort	$N \log N$	N
elementary PQ	$M N$	M
binary heap	$N \log M$	M
best in theory	N	M

order of growth of finding the largest M in a stream of N items

Priority queue: unordered and ordered array implementation

operation	argument	return value	size	contents (unordered)	contents (ordered)
insert	P		1	P	P
insert	Q		2	P Q	P Q
insert	E		3	P Q E	E P Q
remove max		Q	2	P E	E P
insert	X		3	P E X	E P X
insert	A		4	P E X A	A E P X
insert	M		5	P E X A M	A E M P X
remove max		X	4	P E M A	A E M P
insert	P		5	P E M A P	A E M P P
insert	L		6	P E M A P L	A E L M P P
insert	E		7	P E M A P L E	A E E L M P P
remove max		P	6	E M A P L E	A E E L M P

A sequence of operations on a priority queue

Priority queue: implementations cost summary

Challenge. Implement all operations efficiently.

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
goal	$\log N$	$\log N$	$\log N$

order of growth of running time for priority queue with N items

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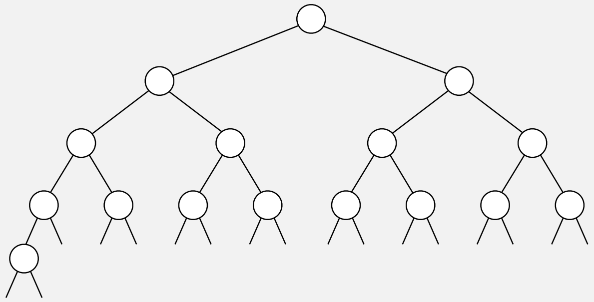
2.4 PRIORITY QUEUES

- ▸ API and elementary implementations
- ▸ binary heaps
- ▸ heapsort
- ▸ event-driven simulation

Complete binary tree

Binary tree. Empty or node with links to left and right binary trees.

Complete tree. Perfectly balanced, except for bottom level.



complete binary tree with N = 16 nodes (height = 4)

Property. Height of complete binary tree with N nodes is $\lfloor \lg N \rfloor$.

Pf. Height increases only when N is a power of 2.

A complete binary tree in nature

Hyphaene Compressa - Doum Palm

© Shlomit Pinter

Binary heap: representation

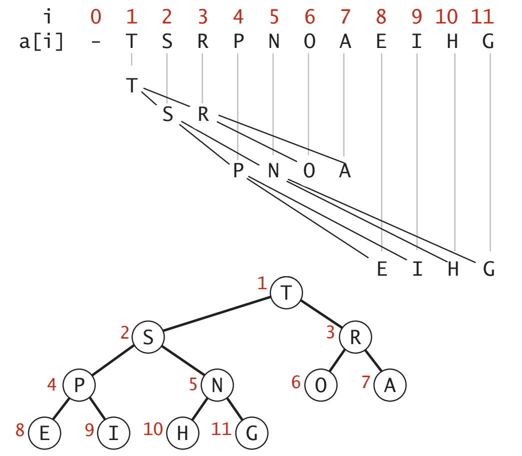
Binary heap. Array representation of a heap-ordered complete binary tree.

Heap-ordered binary tree.

- Keys in nodes.
- Parent's key no smaller than children's keys.

Array representation.

- Indices start at 1.
- Take nodes in level order.
- No explicit links needed!



i	0	1	2	3	4	5	6	7	8	9	10	11
a[i]	-	T	S	R	P	N	O	A	E	I	H	G

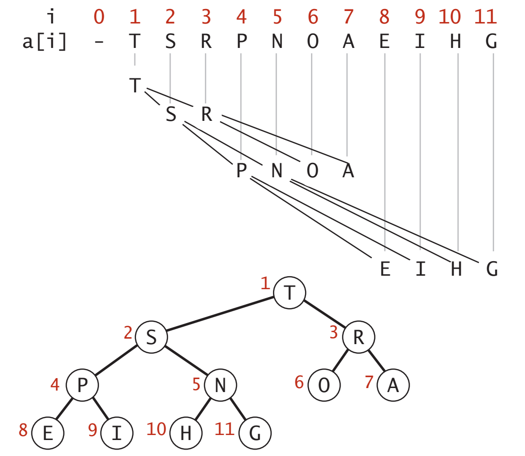
Heap representations

Binary heap: properties

Proposition. Largest key is `a[1]`, which is root of binary tree.

Proposition. Can use array indices to move through tree.

- Parent of node at `k` is at `k/2`.
- Children of node at `k` are at `2k` and `2k+1`.



i	0	1	2	3	4	5	6	7	8	9	10	11
a[i]	-	T	S	R	P	N	O	A	E	I	H	G

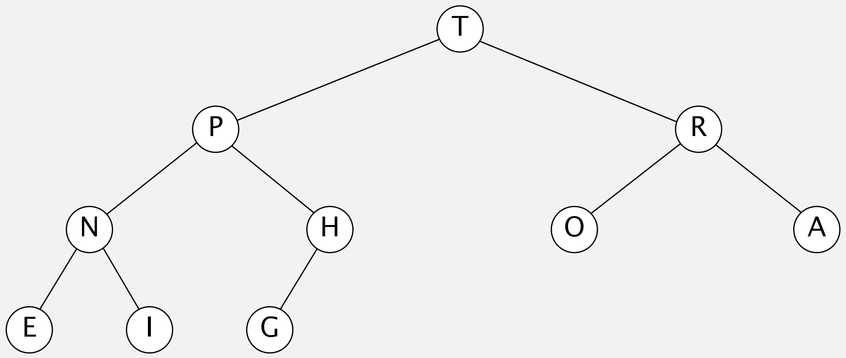
Heap representations

Binary heap demo

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



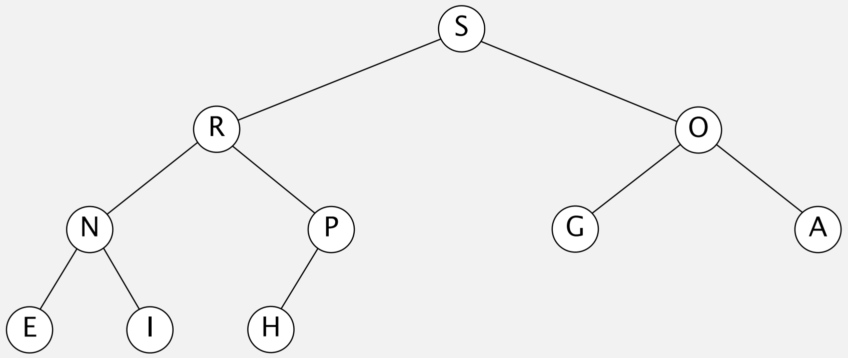
T	P	R	N	H	O	A	E	I	G	

Binary heap demo

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



S	R	O	N	P	G	A	E	I	H	

Binary heap: promotion

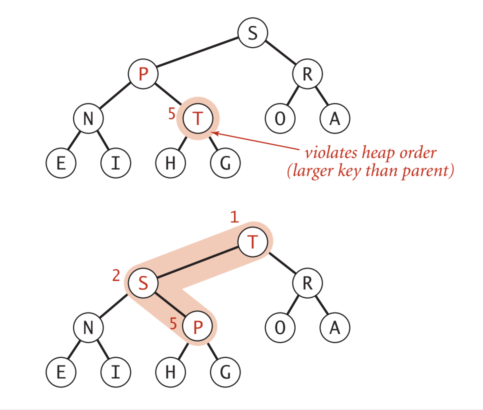
Scenario. A key becomes larger than its parent's key.

To eliminate the violation:

- Exchange key in child with key in parent.
- Repeat until heap order restored.

```
private void swim(int k)
{
   while (k > 1 && less(k/2, k))
   {
      exch(k, k/2);
      k = k/2;
   }
}
```

parent of node at k is at k/2



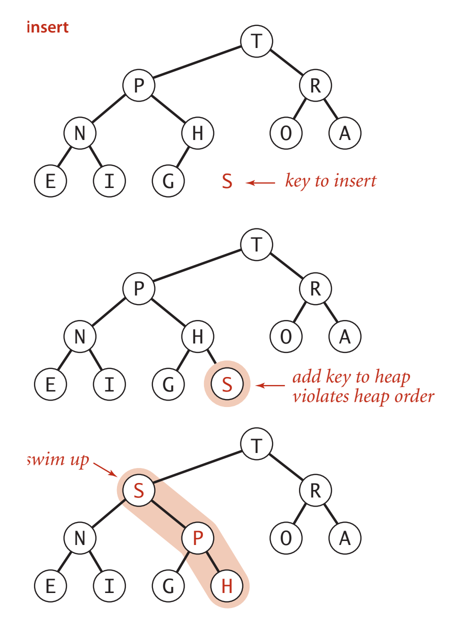
Peter principle. Node promoted to level of incompetence.

Binary heap: insertion

Insert. Add node at end, then swim it up.

Cost. At most $1 + \lg N$ compares.

```
public void insert(Key x)
{
   pq[++N] = x;
   swim(N);
}
```



Binary heap: demotion

Scenario. A key becomes smaller than one (or both) of its children's.

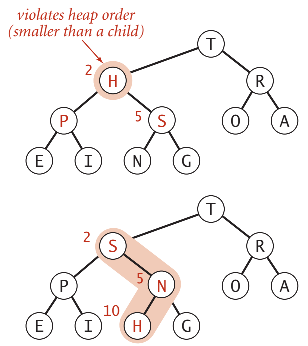
To eliminate the violation:

why not smaller child?

- Exchange key in parent with key in larger child.
- Repeat until heap order restored.

```
private void sink(int k)
{
   while (2*k <= N)
   {
      int j = 2*k;
      if (j < N && less(j, j+1)) j++;
      if (!less(k, j)) break;
      exch(k, j);
      k = j;
   }
}
```

children of node at k are 2k and 2k+1



Top-down reheapify (sink)

Power struggle. Better subordinate promoted.

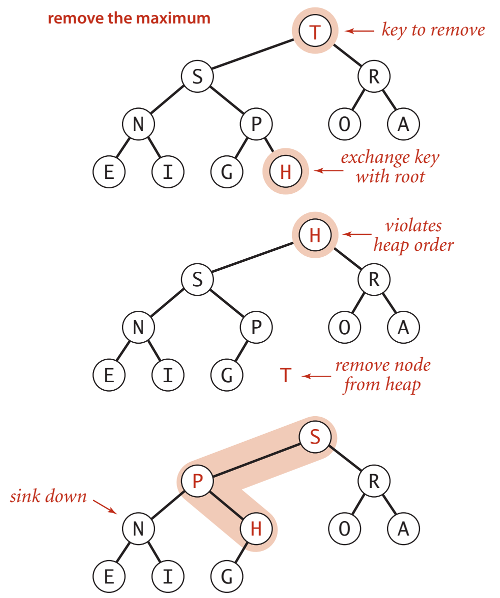
Binary heap: delete the maximum

Delete max. Exchange root with node at end, then sink it down.

Cost. At most $2 \lg N$ compares.

```
public Key delMax()
{
   Key max = pq[1];
   exch(1, N--);
   sink(1);
   pq[N+1] = null;
   return max;
}
```

prevent loitering



Binary heap: Java implementation

```
public class MaxPQ<Key extends Comparable<Key>>
{
   private Key[] pq;
   private int N;

   public MaxPQ(int capacity)
   {  pq = (Key[]) new Comparable[capacity+1];  }

   public boolean isEmpty()
   {  return N == 0;  }
   public void insert(Key key)  // see previous code
   public Key delMax()          // see previous code

   private void swim(int k)     // see previous code
   private void sink(int k)     // see previous code

   private boolean less(int i, int j)
   {  return pq[i].compareTo(pq[j]) < 0;  }
   private void exch(int i, int j)
   {  Key t = pq[i]; pq[i] = pq[j]; pq[j] = t;  }
}
```

fixed capacity
(for simplicity)

PQ ops

heap helper functions

array helper functions

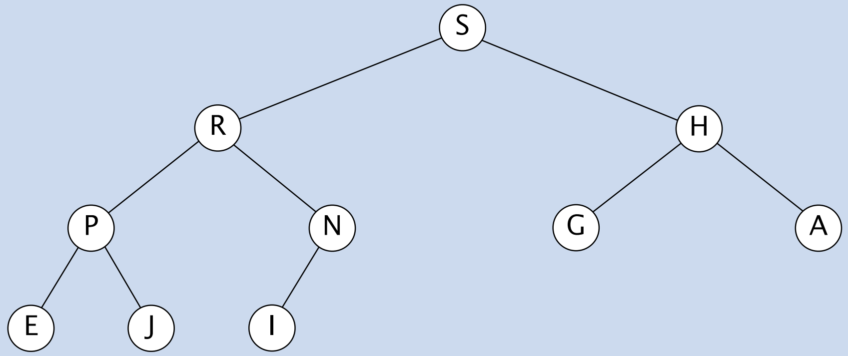
Priority queue: implementations cost summary

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
binary heap	$\log N$	$\log N$	1

order-of-growth of running time for priority queue with N items

DELETE-RANDOM FROM A BINARY HEAP

Goal. Delete a random key from a binary heap in logarithmic time.

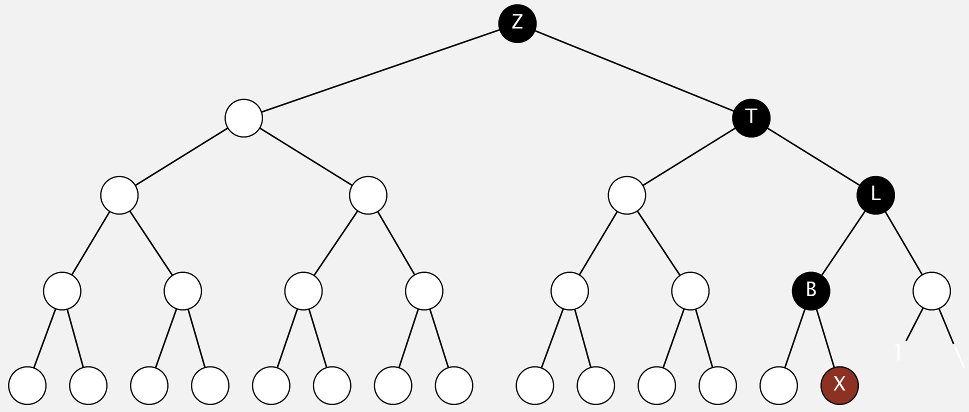


S	R	H	P	N	G	A	E	J	I	

Binary heap: practical improvements

Do "half-exchanges" in sink and swim.

- Reduces number of array accesses.
- Worth doing.

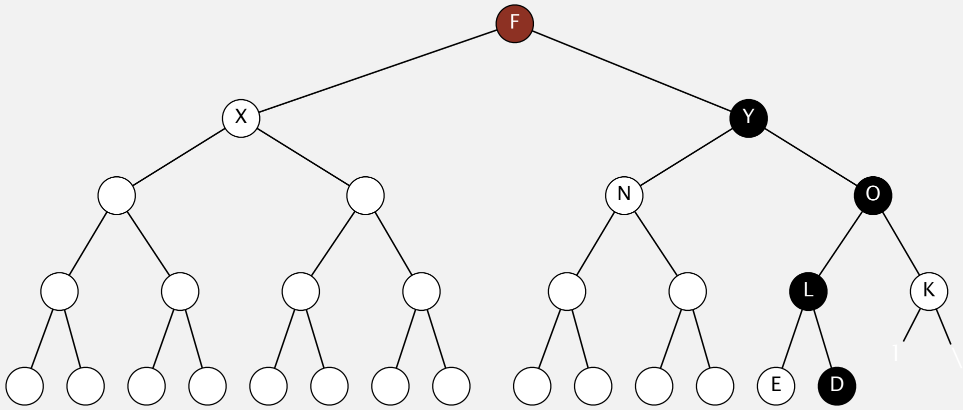


Binary heap: practical improvements

Floyd's "bounce" heuristic.

- Sink key at root all the way to bottom. ⟵ only 1 compare per node
- Swim key back up. ⟵ some extra compares and exchanges
- Overall, fewer compares; more exchanges.
- Worthwhile depending on cost of compare and exchange.

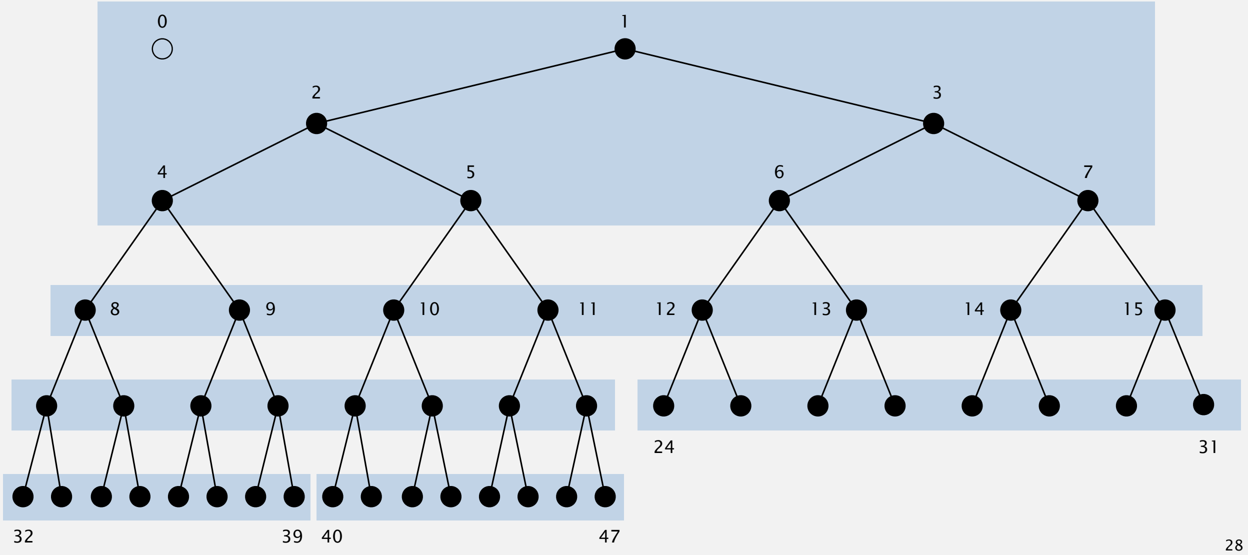
R. W. Floyd
1978 Turing award



Binary heap: practical improvements

Caching. Binary heap is not cache friendly.

Binary heap memory layout (page size = 8 nodes)

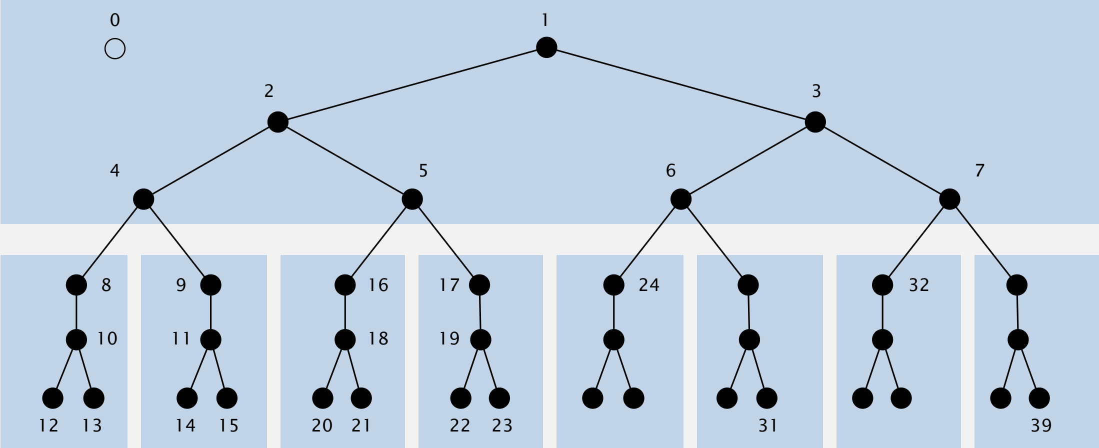


Binary heap: practical improvements

Caching. Binary heap is not cache friendly.

- Cache-aligned d-heap.
- Funnel heap.
- B-heap.
- ...

B-heap memory layout (page size = 8 nodes)

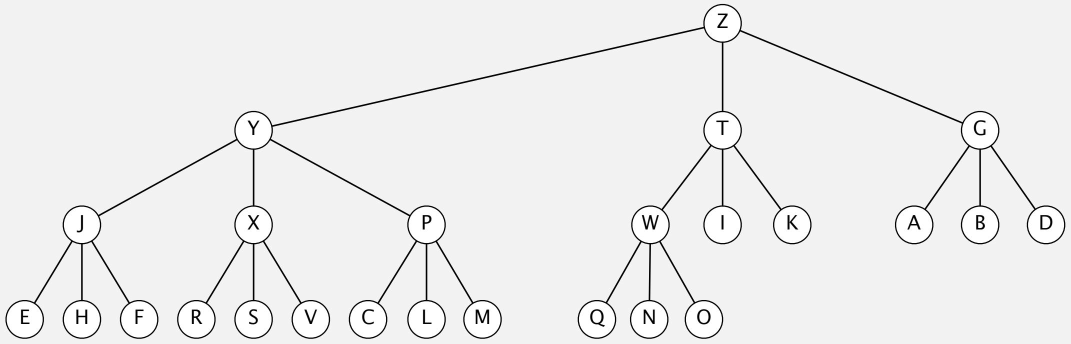


Binary heap: practical improvements

Multiway heaps.

- Complete d-way tree.
- Parent's key no smaller than its children's keys.

Fact. Height of complete d-way tree on N nodes is $\sim \log_d N$.



3-way heap

Priority queues: quiz 1

How many compares (in the worst case) to insert in a d-way heap?

A. $\sim \log_2 N$

B. $\sim \log_d N$

C. $\sim d \log_2 N$

D. $\sim d \log_d N$

E. *I don't know.*

Priority queues: quiz 2

How many compares (in the worst case) to delete-max in a d-way heap?

A. $\sim \log_2 N$

B. $\sim \log_d N$

C. $\sim d \log_2 N$

D. $\sim d \log_d N$

E. *I don't know.*

Priority queue: implementation cost summary

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
binary heap	$\log N$	$\log N$	1
d-ary heap	$\log_d N$	$d \log_d N$	1
Fibonacci	1	$\log N$ †	1
Brodal queue	1	$\log N$	1
impossible	1	1	1

← sweet spot: d = 4

← why impossible?

† amortized

order-of-growth of running time for priority queue with N items

---

Binary heap: considerations

Underflow and overflow.

- Underflow: throw exception if deleting from empty PQ.
- Overflow: add no-arg constructor and use resizing array.

leads to log N amortized time per op (how to make worst case?)

Minimum-oriented priority queue.

- Replace `less()` with `greater()`.
- Implement `greater()`.

Other operations.

- Remove an arbitrary item.
- Change the priority of an item.

can implement efficiently with `sink()` and `swim()`
[stay tuned for Prim/Dijkstra]

Immutability of keys.

- Assumption: client does not change keys while they're on the PQ.
- Best practice: use immutable keys.

Immutability: implementing in Java

Data type. Set of values and operations on those values.
Immutable data type. Can't change the data type value once created.

```
public final class Vector {
   private final int N;
   private final double[] data;

   public Vector(double[] data) {
      this.N = data.length;
      this.data = new double[N];
      for (int i = 0; i < N; i++)
         this.data[i] = data[i];
   }

   …
}
```

← can't override instance methods

← instance variables private and final

← defensive copy of mutable instance variables

← instance methods don't change instance variables

Immutable. `String`, `Integer`, `Double`, `Color`, `Vector`, `Transaction`, `Point2D`.
Mutable. `StringBuilder`, `Stack`, `Counter`, Java array.

Immutability: properties

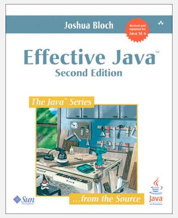
Data type. Set of values and operations on those values.
Immutable data type. Can't change the data type value once created.

Advantages.

- Simplifies debugging.
- Safer in presence of hostile code.
- Simplifies concurrent programming.
- Safe to use as key in priority queue or symbol table.

Disadvantage. Must create new object for each data type value.

> *" Classes should be immutable unless there's a very good reason to make them mutable.… If a class cannot be made immutable, you should still limit its mutability as much as possible. "*
>
> *— Joshua Bloch (Java architect)*



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2.4 PRIORITY QUEUES

- API and elementary implementations
- binary heaps
- *heapsort*
- event-driven simulation

Priority queues: quiz 2

What is the name of this sorting algorithm?

```
public void sort(String[] a)
{
    int N = a.length;
    MaxPQ<String> pq = new MaxPQ<String>();
    for (int i = 0; i < N; i++)
        pq.insert(a[i]);
    for (int i = N-1; i >= 0; i--)
        a[i] = pq.delMax();
}
```

A. Insertion sort.

B. Mergesort.

C. Quicksort.

D. *None of the above.*

E. *I don't know.*

Priority queues: quiz 3

What are its properties?

```
public void sort(String[] a)
{
    int N = a.length;
    MaxPQ<String> pq = new MaxPQ<String>();
    for (int i = 0; i < N; i++)
        pq.insert(a[i]);
    for (int i = N-1; i >= 0; i--)
        a[i] = pq.delMax();
}
```

A. $N \log N$ compares in the worst case.

B. In-place.

C. Stable.

D. *All of the above.*

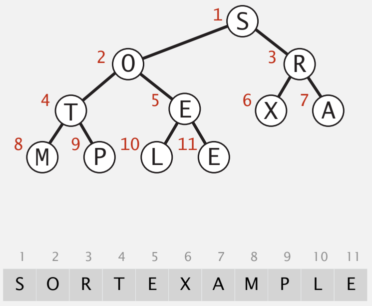
E. *I don't know.*

Heapsort

Basic plan for in-place sort.

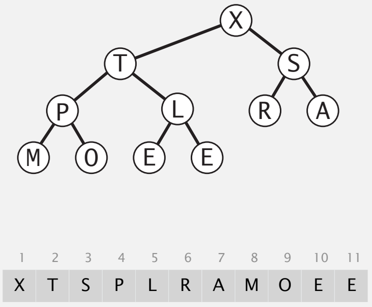
- View input array as a complete binary tree.
- Heap construction: build a max-heap with all N keys.
- Sortdown: repeatedly remove the maximum key.

keys in arbitrary order



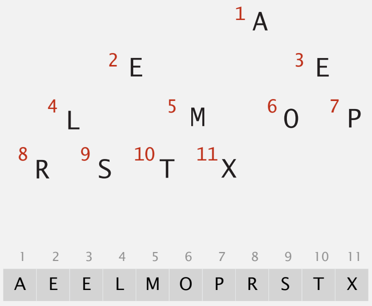
1	2	3	4	5	6	7	8	9	10	11
S	O	R	T	E	X	A	M	P	L	E

build max heap (in place)



1	2	3	4	5	6	7	8	9	10	11
X	T	S	P	L	R	A	M	O	E	E

sorted result (in place)



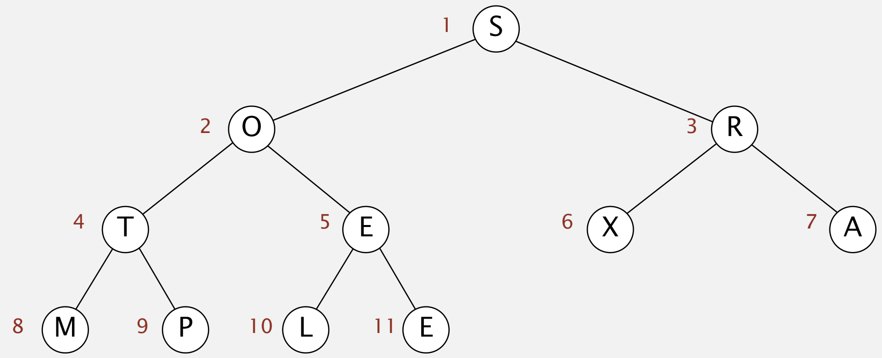
1	2	3	4	5	6	7	8	9	10	11
A	E	E	L	M	O	P	R	S	T	X

Heapsort demo

Heap construction. Build max heap using bottom-up method.

we assume array entries are indexed 1 to N

array in arbitrary order



S	O	R	T	E	X	A	M	P	L	E
1	2	3	4	5	6	7	8	9	10	11

Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.

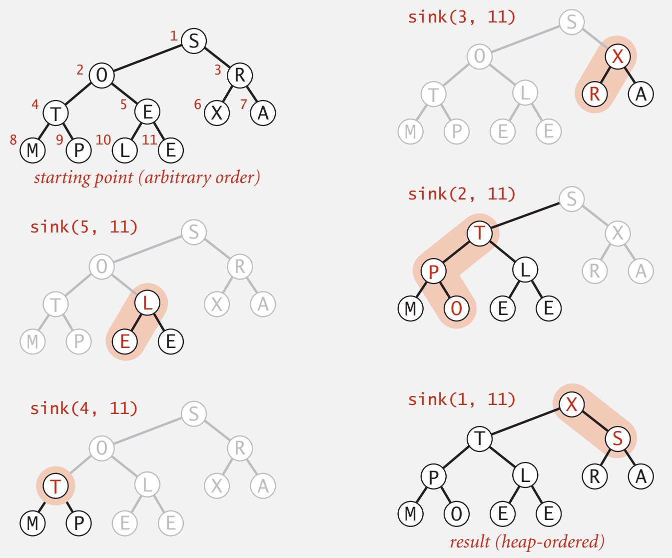
array in sorted order

A	E	E	L	M	O	P	R	S	T	X
1	2	3	4	5	6	7	8	9	10	11

Heapsort: heap construction

First pass. Build heap using bottom-up method.

```
for (int k = N/2; k >= 1; k--)
   sink(a, k, N);
```

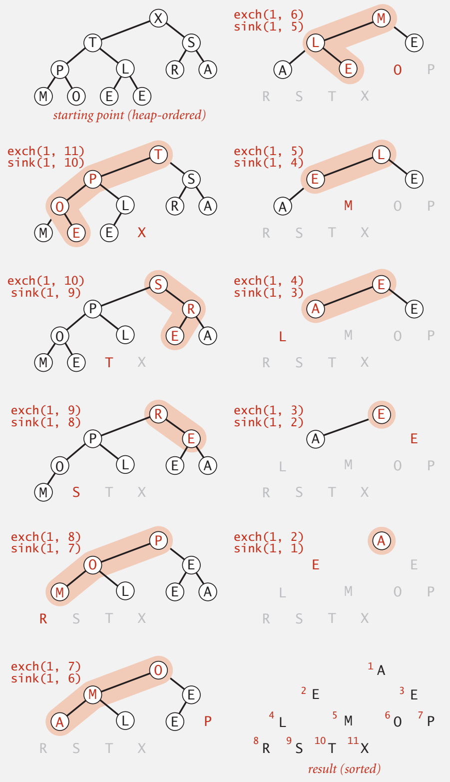


Heapsort: sortdown

Second pass.

- Remove the maximum, one at a time.
- Leave in array, instead of nulling out.

```
while (N > 1)
{
   exch(a, 1, N--);
   sink(a, 1, N);
}
```



Heapsort: Java implementation

```
public class Heap
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int k = N/2; k >= 1; k--)
         sink(a, k, N);
      while (N > 1)
      {
         exch(a, 1, N);
         sink(a, 1, --N);
      }
   }

   private static void sink(Comparable[] a, int k, int N)
   {  /* as before */  }

   private static boolean less(Comparable[] a, int i, int j)
   {  /* as before */  }

   private static void exch(Object[] a, int i, int j)
   {  /* as before */  }

}
```

but make static (and pass arguments)

but convert from 1-based indexing to 0-base indexing

Heapsort: trace

		a[i]											
N	k	0	1	2	3	4	5	6	7	8	9	10	11
initial values			S	O	R	T	E	X	A	M	P	L	E
11	5		S	O	R	T	L	X	A	M	P	E	E
11	4		S	O	R	T	L	X	A	M	P	E	E
11	3		S	O	X	T	L	R	A	M	P	E	E
11	2		S	T	X	P	L	R	A	M	O	E	E
11	1		X	T	S	P	L	R	A	M	O	E	E
heap-ordered			X	T	S	P	L	R	A	M	O	E	E
10	1		T	P	S	O	L	R	A	M	E	E	X
9	1		S	P	R	O	L	E	A	M	E	T	X
8	1		R	P	E	O	L	E	A	M	S	T	X
7	1		P	O	E	M	L	E	A	R	S	T	X
6	1		O	M	E	A	L	E	P	R	S	T	X
5	1		M	L	E	A	E	O	P	R	S	T	X
4	1		L	E	E	A	M	O	P	R	S	T	X
3	1		E	A	E	L	M	O	P	R	S	T	X
2	1		E	A	E	L	M	O	P	R	S	T	X
1	1		A	E	E	L	M	O	P	R	S	T	X
sorted result			A	E	E	L	M	O	P	R	S	T	X

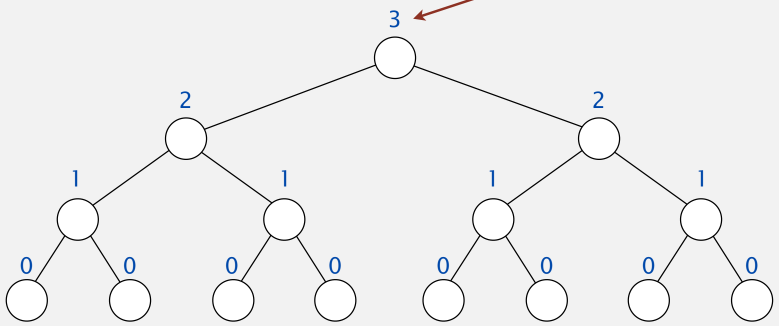
Heapsort trace (array contents just after each sink)

Heapsort: mathematical analysis

Proposition. Heap construction uses ≤ $2N$ compares and ≤ N exchanges.

Pf sketch. [assume $N = 2^{h+1} - 1$]

max number of exchanges to sink node



binary heap of height h = 3

a tricky sum (see COS 340)

$$h + 2(h-1) + 4(h-2) + 8(h-3) + \ldots + 2^h(0) \le 2^{h+1} - 1$$
$$= N$$

Heapsort: mathematical analysis

Proposition. Heap construction uses ≤ $2N$ compares and ≤ N exchanges.

Proposition. Heapsort uses ≤ $2N \lg N$ compares and exchanges.

algorithm can be improved to ~ 1 N lg N (but no such variant is known to be practical)

Significance. In-place sorting algorithm with $N \log N$ worst-case.

- Mergesort: no, linear extra space. ← in-place merge possible, not practical
- Quicksort: no, quadratic time in worst case. ← N log N worst-case quicksort possible, not practical
- Heapsort: yes!

Bottom line. Heapsort is optimal for both time and space, but:

- Inner loop longer than quicksort's.
- Makes poor use of cache.
- Not stable.

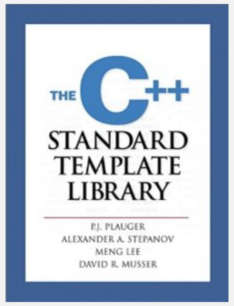
can be improved using advanced caching tricks

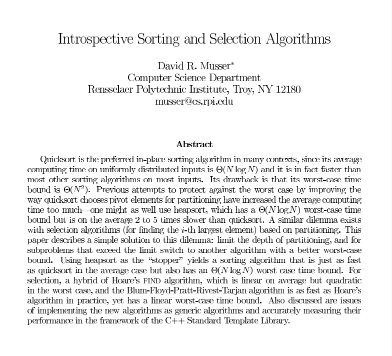
Introsort

Goal. As fast as quicksort in practice; $N \log N$ worst case, in place.

Introsort.

- Run quicksort.
- Cutoff to heapsort if stack depth exceeds $2 \lg N$.
- Cutoff to insertion sort for $N = 16$.





Introspective Sorting and Selection Algorithms

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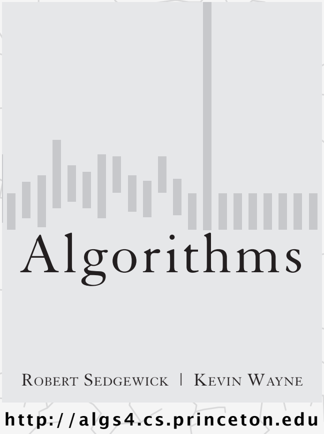
Abstract

Quicksort is the preferred in-place sorting algorithm in many contexts, since its average computing time on uniformly distributed inputs is $\Theta(N \log N)$ and it is in fact faster than most other sorting algorithms on most inputs. Its drawback is that its worst-case time bound is $\Theta(N^2)$. Previous attempts to protect against the worst case by improving the way quicksort chooses pivot elements for partitioning have increased the average computing time too much—one might as well use heapsort, which has a $\Theta(N \log N)$ worst-case time bound but is on the average 2 to 5 times slower than quicksort. A similar dilemma exists with selection algorithms (for finding the i-th largest element) based on partitioning. This paper describes a simple solution to this dilemma: limit the depth of partitioning, and for subproblems that exceed the limit switch to another algorithm with a better worst-case bound. Using heapsort as the "stopper" yields a sorting algorithm that is just as fast as quicksort in the average case but also has an $\Theta(N \log N)$ worst case time bound. For selection, a hybrid of Hoare's FIND algorithm, which is linear on average but quadratic in the worst case, and the Blum-Floyd-Pratt-Rivest-Tarjan algorithm is as fast as Hoare's algorithm in practice, yet has a linear worst-case time bound. Also discussed are issues of implementing the new algorithms as generic algorithms and accurately measuring their performance in the framework of the C++ Standard Template Library.

In the wild. C++ STL, Microsoft .NET Framework.

Sorting algorithms: summary

	inplace?	stable?	best	average	worst	remarks
selection	✔		$½ N^2$	$½ N^2$	$½ N^2$	N exchanges
insertion	✔	✔	N	$¼ N^2$	$½ N^2$	use for small N or partially ordered
shell	✔		$N \log_3 N$	?	$c N^{3/2}$	tight code; subquadratic
merge		✔	$½ N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee; stable
timsort		✔	N	$N \lg N$	$N \lg N$	improves mergesort when preexisting order
quick	✔		$N \lg N$	$2 N \ln N$	$½ N^2$	$N \log N$ probabilistic guarantee; fastest in practice
3-way quick	✔		N	$2 N \ln N$	$½ N^2$	improves quicksort when duplicate keys
heap	✔		N	$2 N \lg N$	$2 N \lg N$	$N \log N$ guarantee; in-place
?	✔	✔	N	$N \lg N$	$N \lg N$	holy sorting grail

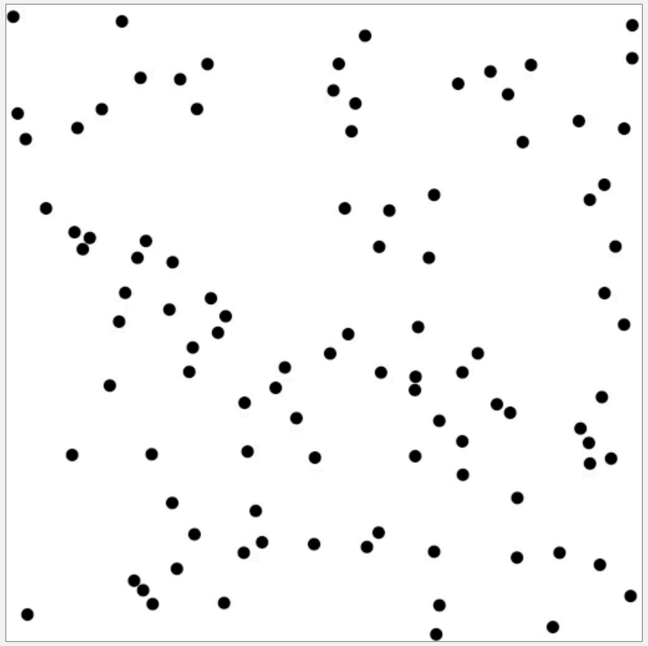


2.4 Priority Queues

- API and elementary implementations
- binary heaps
- heapsort
- *event-driven simulation*

Molecular dynamics simulation of hard discs

Goal. Simulate the motion of N moving particles that behave according to the laws of elastic collision.



Molecular dynamics simulation of hard discs

Goal. Simulate the motion of N moving particles that behave according to the laws of elastic collision.

Hard disc model.

- Moving particles interact via elastic collisions with each other and walls.
- Each particle is a disc with known position, velocity, mass, and radius.
- No other forces.

temperature, pressure, diffusion constant

motion of individual atoms and molecules

Significance. Relates macroscopic observables to microscopic dynamics.

- Maxwell-Boltzmann: distribution of speeds as a function of temperature.
- Einstein: explain Brownian motion of pollen grains.

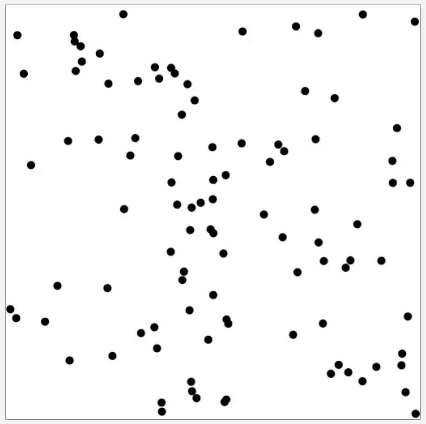
Warmup: bouncing balls

Time-driven simulation. N bouncing balls in the unit square.

```
public class BouncingBalls
{
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      Ball[] balls = new Ball[N];
      for (int i = 0; i < N; i++)
         balls[i] = new Ball();
      while(true)
      {
         StdDraw.clear();
         for (int i = 0; i < N; i++)
         {
            balls[i].move(0.5);
            balls[i].draw();
         }
         StdDraw.show(50);
      }
   }
}
```

main simulation loop

```
% java BouncingBalls 100
```



Warmup: bouncing balls

```
public class Ball
{
    private double rx, ry;        // position
    private double vx, vy;        // velocity
    private final double radius;  // radius
    public Ball(...)
    {  /* initialize position and velocity */  }

    public void move(double dt)
    {
        if ((rx + vx*dt < radius) || (rx + vx*dt > 1.0 - radius)) { vx = -vx; }
        if ((ry + vy*dt < radius) || (ry + vy*dt > 1.0 - radius)) { vy = -vy; }
        rx = rx + vx*dt;
        ry = ry + vy*dt;
    }
    public void draw()
    {  StdDraw.filledCircle(rx, ry, radius);  }
}
```

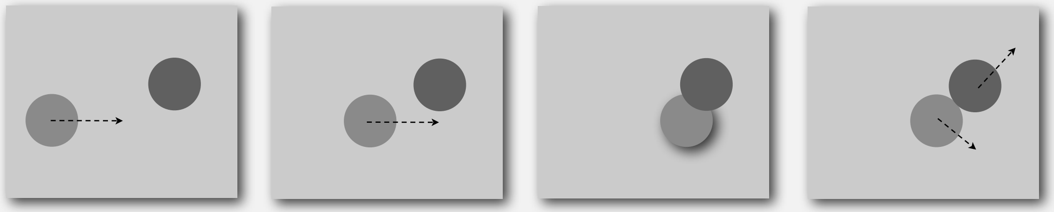
check for collision with walls

Missing. Check for balls colliding with each other.

- Physics problems: when? what effect?
- CS problems: which object does the check? too many checks?

Time-driven simulation

- Discretize time in quanta of size dt.
- Update the position of each particle after every dt units of time, and check for overlaps.
- If overlap, roll back the clock to the time of the collision, update the velocities of the colliding particles, and continue the simulation.

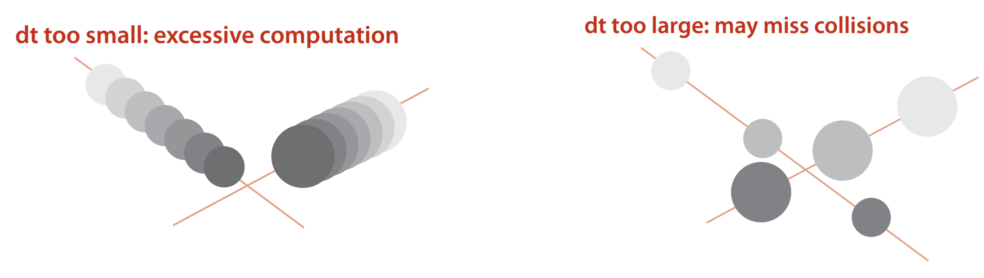


t **t + dt** **t + 2 dt (collision detected)** **t + Δt (roll back clock)**

Time-driven simulation

Main drawbacks.

- ~ $N^2 / 2$ overlap checks per time quantum.
- Simulation is too slow if dt is very small.
- May miss collisions if dt is too large.
  (if colliding particles fail to overlap when we are looking)



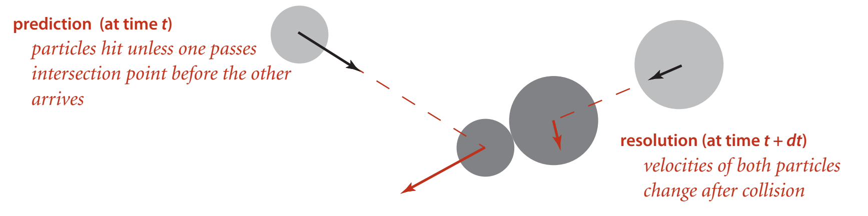
Event-driven simulation

Change state only when something happens.

- Between collisions, particles move in straight-line trajectories.
- Focus only on times when collisions occur.
- Maintain PQ of collision events, prioritized by time.
- Remove the min = get next collision.

Collision prediction. Given position, velocity, and radius of a particle, when will it collide next with a wall or another particle?

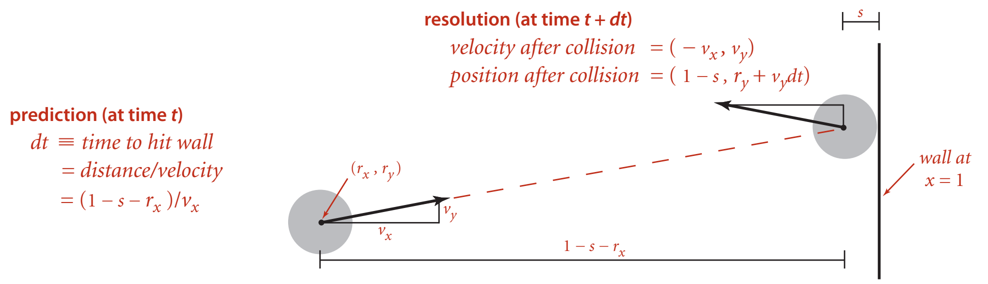
Collision resolution. If collision occurs, update colliding particle(s) according to laws of elastic collisions.



Particle-wall collision

Collision prediction and resolution.

- Particle of radius s at position (rx, ry).
- Particle moving in unit box with velocity (vx, vy).
- Will it collide with a vertical wall? If so, when?

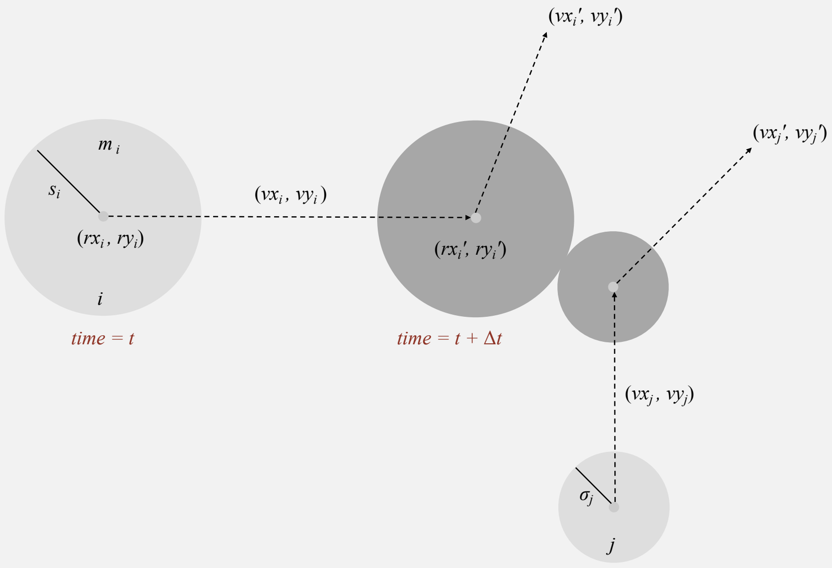


Predicting and resolving a particle-wall collision

Particle-particle collision prediction

Collision prediction.

- Particle i: radius s_i, position (rx_i, ry_i), velocity (vx_i, vy_i).
- Particle j: radius s_j, position (rx_j, ry_j), velocity (vx_j, vy_j).
- Will particles i and j collide? If so, when?



Particle-particle collision prediction

Collision prediction.

- Particle i: radius s_i, position (rx_i, ry_i), velocity (vx_i, vy_i).
- Particle j: radius s_j, position (rx_j, ry_j), velocity (vx_j, vy_j).
- Will particles i and j collide? If so, when?

$$\Delta t = \begin{cases} \infty & \text{if } \Delta v \cdot \Delta r \geq 0, \\ \infty & \text{if } d < 0, \\ -\dfrac{\Delta v \cdot \Delta r \ + \ \sqrt{d}}{\Delta v \cdot \Delta v} & \text{otherwise} \end{cases}$$

$$d \ = \ (\Delta v \cdot \Delta r)^2 \ - \ (\Delta v \cdot \Delta v)\ (\Delta r \cdot \Delta r \ - \ s^2), \qquad s = s_i + s_j$$

$$\Delta v = (\Delta vx,\ \Delta vy) \ = \ (vx_i - vx_j,\ vy_i - vy_j)$$
$$\Delta r = (\Delta rx,\ \Delta ry) \ = \ (rx_i - rx_j,\ ry_i - ry_j)$$

$$\Delta v \cdot \Delta v = (\Delta vx)^2 + \ (\Delta vy)^2$$
$$\Delta r \cdot \Delta r = (\Delta rx)^2 + \ (\Delta ry)^2$$
$$\Delta v \cdot \Delta r = (\Delta vx)(\Delta rx) + \ (\Delta vy)(\Delta ry)$$

Important note: This is physics, so we won't be testing you on it!

Particle-particle collision resolution

Collision resolution. When two particles collide, how does velocity change?

$$\begin{aligned} vx_i' &= vx_i + Jx \,/\, m_i \\ vy_i' &= vy_i + Jy \,/\, m_i \\ vx_j' &= vx_j - Jx \,/\, m_j \\ vy_j' &= vy_j - Jy \,/\, m_j \end{aligned}$$

Newton's second law
(momentum form)

$$Jx = \frac{J\,\Delta rx}{s}, \quad Jy = \frac{J\,\Delta ry}{s}, \quad J = \frac{2\,m_i\,m_j\,(\Delta v \cdot \Delta r)}{s\,(m_i + m_j)}$$

impulse due to normal force
(conservation of energy, conservation of momentum)

Important note: This is physics, so we won't be testing you on it!

Particle data type skeleton

```
public class Particle
{
    private double rx, ry;       // position
    private double vx, vy;       // velocity
    private final double radius; // radius
    private final double mass;   // mass
    private int count;           // number of collisions

    public Particle(...) { }

    public void move(double dt) { }
    public void draw()          { }

    public double timeToHit(Particle that)   { }
    public double timeToHitVerticalWall()    { }
    public double timeToHitHorizontalWall()  { }

    public void bounceOff(Particle that)     { }
    public void bounceOffVerticalWall()      { }
    public void bounceOffHorizontalWall()    { }

}
```

predict collision
with particle or wall

resolve collision
with particle or wall

Particle-particle collision and resolution implementation

```
public double timeToHit(Particle that)
{
   if (this == that) return INFINITY;
   double dx  = that.rx - this.rx, dy  = that.ry - this.ry;
   double dvx = that.vx - this.vx; dvy = that.vy - this.vy;
   double dvdr = dx*dvx + dy*dvy;
   if( dvdr > 0) return INFINITY;
   double dvdv = dvx*dvx + dvy*dvy;
   double drdr = dx*dx + dy*dy;
   double s = this.radius + that.radius;
   double d = (dvdr*dvdr) - dvdv * (drdr - s*s);
   if (d < 0) return INFINITY;
   return -(dvdr + Math.sqrt(d)) / dvdv;
}
```

no collision

```
public void bounceOff(Particle that)
{
   double dx  = that.rx - this.rx, dy  = that.ry - this.ry;
   double dvx = that.vx - this.vx, dvy = that.vy - this.vy;
   double dvdr = dx*dvx + dy*dvy;
   double s = this.radius + that.radius;
   double J = 2 * this.mass * that.mass * dvdr / (s * (this.mass + that.mass));
   double Jx = J * dx / s;
   double Jy = J * dy / s;
   this.vx += Jx / this.mass;
   this.vy += Jy / this.mass;
   that.vx -= Jx / that.mass;
   that.vy -= Jy / that.mass;
   this.count++;
   that.count++;
}
```

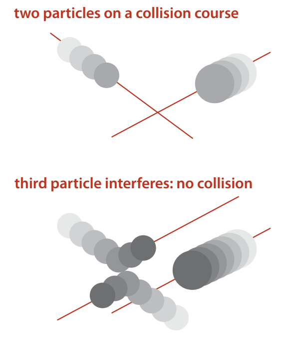
Important note: This is physics, so we won't be testing you on it!

Collision system: event-driven simulation main loop

Initialization.

- Fill PQ with all potential particle-wall collisions.
- Fill PQ with all potential particle-particle collisions.

"potential" since collision may not happen if some other collision intervenes



An invalidated event

Main loop.

- Delete the impending event from PQ (min priority = t).
- If the event has been invalidated, ignore it.
- Advance all particles to time t, on a straight-line trajectory.
- Update the velocities of the colliding particle(s).
- Predict future particle-wall and particle-particle collisions involving the colliding particle(s) and insert events onto PQ.

Event data type

Conventions.

- Neither particle `null` ⇒ particle-particle collision.
- One particle `null` ⇒ particle-wall collision.
- Both particles `null` ⇒ redraw event.

```
private class Event implements Comparable<Event>
{
   private double time;         // time of event
   private Particle a, b;       // particles involved in event
   private int countA, countB;  // collision counts for a and b

   public Event(double t, Particle a, Particle b) { ... }

   public int compareTo(Event that)
   {  return this.time - that.time;   }

   public boolean isValid() { ... }
}
```

create event

ordered by time

invalid if intervening collision

Collision system implementation: skeleton

```
public class CollisionSystem
{
   private MinPQ<Event> pq;        // the priority queue
   private double t  = 0.0;        // simulation clock time
   private Particle[] particles;   // the array of particles

   public CollisionSystem(Particle[] particles) { }

   private void predict(Particle a)
   {
      if (a == null) return;
      for (int i = 0; i < N; i++)
      {
         double dt = a.timeToHit(particles[i]);
         pq.insert(new Event(t + dt, a, particles[i]));
      }
      pq.insert(new Event(t + a.timeToHitVerticalWall()   , a, null));
      pq.insert(new Event(t + a.timeToHitHorizontalWall(), null, a));
   }

   private void redraw()  { }

   public void simulate() {  /* see next slide */  }
}
```

add to PQ all particle-wall and particle-particle collisions involving this particle

---

Collision system implementation: main event-driven simulation loop

```
public void simulate()
{
   pq = new MinPQ<Event>();
   for(int i = 0; i < N; i++) predict(particles[i]);
   pq.insert(new Event(0, null, null));

   while(!pq.isEmpty())
   {
      Event event = pq.delMin();
      if(!event.isValid()) continue;
      Particle a = event.a;
      Particle b = event.b;

      for(int i = 0; i < N; i++)
         particles[i].move(event.time - t);
      t = event.time;

      if      (a != null && b != null) a.bounceOff(b);
      else if (a != null && b == null) a.bounceOffVerticalWall()
      else if (a == null && b != null) b.bounceOffHorizontalWall();
      else if (a == null && b == null) redraw();

      predict(a);
      predict(b);
   }
}
```

initialize PQ with collision events and redraw event

get next event

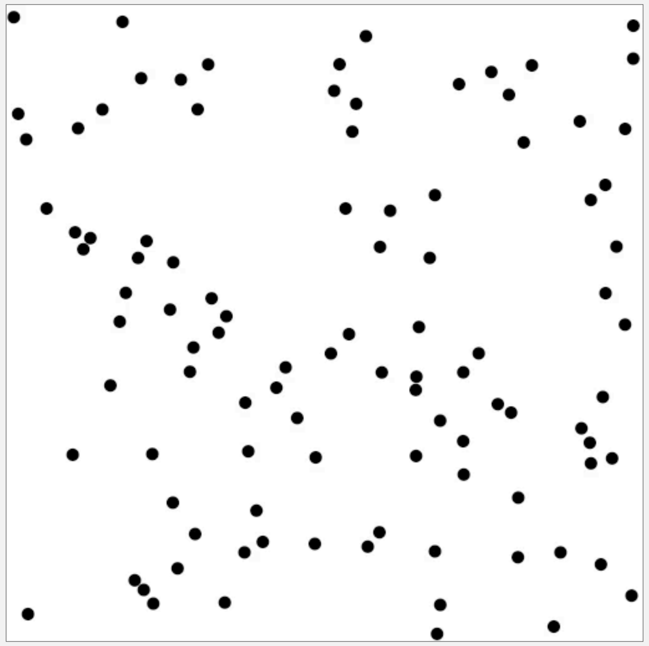
update positions and time

process event

predict new events based on changes

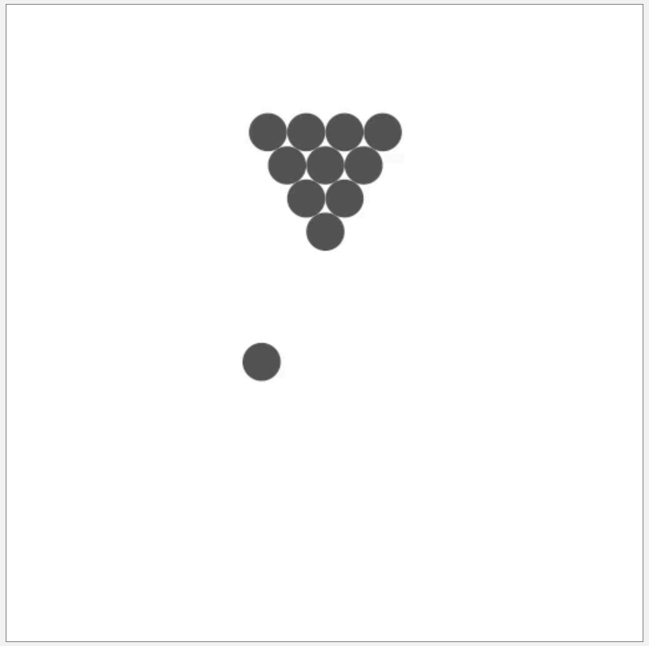
Particle collision simulation: example 1

```
% java CollisionSystem 100
```



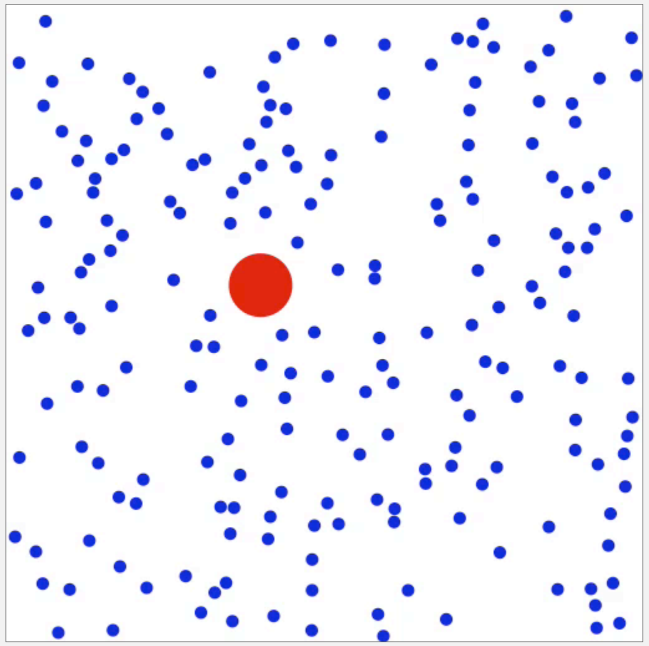
Particle collision simulation: example 2

```
% java CollisionSystem < billiards.txt
```



Particle collision simulation: example 3

```
% java CollisionSystem < brownian.txt
```



Particle collision simulation: example 4

```
% java CollisionSystem < diffusion.txt
```