Finite Difference Approximations
For Derivatives
Taylor Series

- Goal: given smooth function $f : \mathbb{R} \to \mathbb{R}$, find approximate derivatives at some point $x$
- Consider Taylor series expansions around $x$:

\[
f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2} h^2 + \frac{f'''(x)}{2} h^3 + \ldots
\]
\[
f(x - h) = f(x) - f'(x)h + \frac{f''(x)}{2} h^2 - \frac{f'''(x)}{2} h^3 + \ldots
\]
Forward Difference

- Starting from first equation,

\[
f(x + h) \approx f(x) + f'(x)h + O(h^2)
\]

\[
f'(x) \approx \frac{f(x + h) - f(x)}{h} + O(h)
\]

- This is the forward-difference approximation to the first derivative: first-order accurate
• Similarly, starting from second equation,

\[ f(x - h) \approx f(x) - f'(x)h + O(h^2) \]

\[ f'(x) \approx \frac{f(x) - f(x - h)}{h} + O(h) \]

• This is the \textit{backward-difference} approximation to the first derivative: also first-order accurate
Centered Difference

• Subtract the two Taylor-series expansions:

\[
f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2} h^2 + \frac{f'''(x)}{2} h^3 + \ldots
\]

\[
f(x - h) = f(x) - f'(x)h + \frac{f''(x)}{2} h^2 - \frac{f'''(x)}{2} h^3 + \ldots
\]

\[
f(x + h) - f(x - h) \approx 2f'(x)h + O(h^3)
\]

\[
f'(x) \approx \frac{f(x + h) - f(x - h)}{2h} + O(h^2)
\]

• This is the centered-difference approximation to the first derivative: second-order accurate
Second Derivative

• Now add the two Taylor-series expansions:

\[ f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{2}h^3 + \ldots \]

\[ f(x - h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{2}h^3 + \ldots \]

\[ f(x + h) + f(x - h) \approx 2f(x) + f''(x)h^2 + O(h^4) \]

\[ f''(x) \approx \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} + O(h^2) \]

• This is the centered-difference approximation to the second derivative: second-order accurate