Chaos
Lorenz Equations

\[
\frac{dx}{dt} = -\sigma x + \sigma y
\]

\[
\frac{dy}{dt} = rx - y - xz
\]

\[
\frac{dz}{dt} = -bz + xy
\]
x over time for 2 initial conditions
Logistic Map

• Verhulst equation:

\[ \frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) \]

• Logistic map:

\[ x_{n+1} = 4rx_n (1 - x_n) \]

Maps \([0,1] \rightarrow [0,1]\)
Logistic map: $r = 0.240, x_0 = 0.100$
Logistic map: $r = 0.8700$, $x_0 = 0.100$
Logistic map: $r = 0.90$, $x_0 = 0.100$
Iterated Logistic Map Demo
http://ibiblio.org/e-notes/MSet/Logistic.htm
Bifurcation diagram

Periods of order!

Bifurcation

Chaos

Bifurcation diagram
Lyapunov exponent – how quickly do solutions diverge under perturbation?

- Period doubling
- Super-stable trajectories
- Chaos
Doubling route to chaos

Intervals between doublings get smaller and smaller. The limit \( \delta = \lim_{k \to \infty} \frac{\delta_k}{\delta_{k+1}} \) is known as Feigenbaum’s constant.

- \( \delta = 4.669201609102990671853203821578 \ldots \)
- Independent of shape of map, as long as there’s a simple quadratic maximum
- Universal “route to chaos”: examples in electrical circuits (ODEs), water flow (PDEs), …
Iterated logistic map

Economic applications: see Medio 92, Puu 03
Corn-Hog cycle:

Corn-Hog cycle (William King, Drexel)