COS 323, Fall 2012
Exam 2

Name:

NetID:

Honor Code pledge:

Signature:

- This exam consists of 5 questions on 7 pages.
- Do all of your work on these pages, using the back for scratch space and giving the answers in the spaces provided.
- No partial credit will be provided unless you show your work and/or explain your reasoning.
- This is a closed-book, closed-notes exam.
- **Put your NetID on every page (1 point), and write out and sign the Honor Code pledge before turning in the test:**

  "I pledge my honor that I have not violated the Honor Code during this examination."

Question	1	2	3	4	5	NetID on each page	**Total**
Score							

1. Numerical Integration

(a) Consider computing the definite integral

$$\int_0^2 x^3\,dx$$

using the trapezoidal rule. For a step size $h = {}^1\!/_2$, this yields an error of ${}^1\!/_4$. How big should the interval h be to get an error less than or equal to ${}^1\!/_{64}$?

(b) Now consider computing the same integral using Simpson's rule. How big should the step size h be to get an error less than or equal to ${}^1\!/_{64}$? ***(Warning: this may be considered a trick question.)***

2. Richardson extrapolation

We saw that, for the trapezoidal rule, one can combine the estimated integral for step size h with the estimated integral for step size $h/2$ as follows:

$$\tfrac{4}{3}F_{h/2} - \tfrac{1}{3}F_h$$

to obtain a better estimate that is fourth-order accurate in h.

We also saw that for the midpoint method, it is natural to cut the interval into ***thirds*** when reducing the step size, as opposed to cutting it in half.

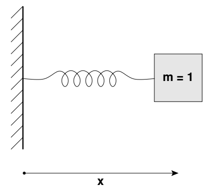
Derive the expression for Richardson extrapolation applied to the midpoint method. Specifically, given the estimated integral for step size h, called F_h, and the estimated integral for step size $h/3$, called $F_{h/3}$, what linear combination of them yields a fourth-order estimate of the integral?

3. Differential equation true/false

(a) Because the Backward Euler method is stable for arbitrary step sizes, it is always the best choice for solving ODEs. ***True or false? Why or why not?***

(b) In using a fully discrete finite difference method for solving a time-dependent PDE, the sizes of the time step and space step can always be chosen independently of each other. ***True or false? Why or why not?***

4. Solving ODEs



Consider simulating a 1-dimensional mass-spring system, with a particle of mass $m = 1$ attached to a frictionless spring with rest length $l = 0$ and spring constant $k = 1$. The other end of the spring is fixed at $x = 0$. This leads to a second-order differential equation:

$$\frac{d^2x}{dt^2} = -x$$

Assume that the initial conditions are:

Initial position $x_0 = 1$
Initial velocity $v_0 = 0$

(a) Transform the second-order differential equation into a system of first-order differential equations. Use v as the additional variable.

(b) Compute the position and velocity after each of two iterations of Euler's method with a time step of $\Delta t = {}^1\!/_2$. Please write all results as fractions rather than decimals.

b) What is the total energy in the system at time $t = 0$, and after each time step of Euler? Recall that kinetic energy $KE = {}^1\!/_2 mv^2$ while a spring's potential energy $PE = {}^1\!/_2 k(x - l)^2$.

c) Now simulate two iterations of a different method, in which the ***new, updated*** velocity is used to compute the position at each timestep, rather than the velocity from the previous timestep. (This is a special case of an ODE solver called "leapfrog".) Compute the energy in the system, and comment on the expected stability of the two solution methods for simulating mass-spring systems.

5. Base-3 FFT

The FFT algorithm we saw in class recursively performed two FFTs of half the size. As a result, it works only for lengths that are powers of two. However, the same idea generalizes for arbitrary equal subdivisions of the data.

Concretely, consider an FFT algorithm that performs ***three*** recursive FFTs on subsets of the data that are exactly ***one-third*** the length, then combines the results together.

Complete the following equation describing how the sub-FFTs are combined. The summations should each represent Fourier Transforms of length $n/3$. Use the back of this page for derivations, and fill in the answers in the provided boxes.

$$F_k = \sum_{x=0}^{n-1} f_x \, \mathrm{e}^{-2\pi i x k/n}$$

$$= \sum_{x=0}^{n/3-1} f_{\boxed{\qquad}} \, \mathrm{e}^{\boxed{\qquad}} +$$

$$\boxed{\qquad} \sum_{x=0}^{n/3-1} f_{\boxed{\qquad}} \, \mathrm{e}^{\boxed{\qquad}} +$$

$$\boxed{\qquad} \sum_{x=0}^{n/3-1} f_{\boxed{\qquad}} \, \mathrm{e}^{\boxed{\qquad}}$$