Algorithms

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5.3 Substring Search

- introduction
- brute force
- Knuth-Morris-Pratt
- Boyer-Moore
- Rabin-Karp

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- introduction
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Substring search

Goal. Find pattern of length M in a text of length N.

typically N >> M

pattern → N E E D L E

text → I N A H A Y S T A C K N E E D L E I N A

match

Substring search applications

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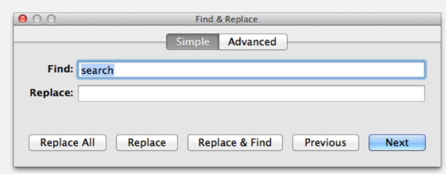


Substring search applications

Goal. Find pattern of length M in a text of length N.

typically N >> M

```
pattern → N E E D L E
text    → I N A H A Y S T A C K N E E D L E I N A
                                ‾‾‾‾‾‾‾‾‾‾‾
                                   match
```

Computer forensics. Search memory or disk for signatures, e.g., all URLs or RSA keys that the user has entered.

http://citp.princeton.edu/memory

Substring search applications

Goal. Find pattern of length M in a text of length N.

typically N >> M

```
pattern → N E E D L E
text    → I N A H A Y S T A C K N E E D L E I N A
                                ‾‾‾‾‾‾‾‾‾‾‾
                                   match
```

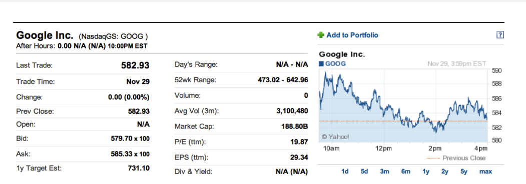
Identify patterns indicative of spam.

- PROFITS
- LOSE WE1GHT
- herbal Viagra
- There is no catch.
- This is a one-time mailing.
- This message is sent in compliance with spam regulations.

Substring search applications

Screen scraping. Extract relevant data from web page.

Ex. Find string delimited by and after first occurrence of pattern Last Trade:.



http://finance.yahoo.com/q?s=goog

```
...
<tr>
<td class= "yfnc_tablehead1"
width= "48%">
Last Trade:
</td>
<td class= "yfnc_tabledata1">
<big><b>452.92</b></big>
</td></tr>
<td class= "yfnc_tablehead1"
width= "48%">
Trade Time:
</td>
<td class= "yfnc_tabledata1">
...
```

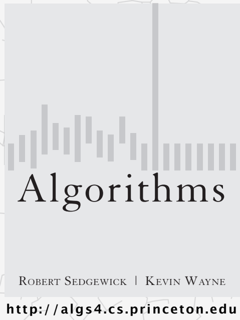

Screen scraping: Java implementation

Java library. The `indexOf()` method in Java's string library returns the index of the first occurrence of a given string, starting at a given offset.

```java
public class StockQuote
{
   public static void main(String[] args)
   {
      String name = "http://finance.yahoo.com/q?s=";
      In in = new In(name + args[0]);
      String text = in.readAll();
      int start    = text.indexOf("Last Trade:", 0);
      int from     = text.indexOf("<b>", start);
      int to       = text.indexOf("</b>", from);
      String price = text.substring(from + 3, to);
      StdOut.println(price);
   }
}
```

```
% java StockQuote goog
582.93

% java StockQuote msft
24.84
```



5.3 Substring Search

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- **brute force**
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Brute-force substring search

Check for pattern starting at each text position.

i	j	i+j	0	1	2	3	4	5	6	7	8	9	10
		txt →	A	B	A	C	A	D	A	B	R	A	C
0	2	2	A	B	R	A ← pat							
1	0	1		A	B	R	A						
2	1	3			A	B	R	A					
3	0	3				A	B	R	A				
4	1	5					A	B	R	A			
5	0	5						A	B	R	A		
6	4	10							A	B	R	A	

entries in red are mismatches

entries in gray are for reference only

entries in black match the text

return `i` *when* `j` *is* `M`

match

Brute-force substring search: Java implementation

Check for pattern starting at each text position.

i	j	i+j	0	1	2	3	4	5	6	7	8	9	10
			A	B	A	C	A	D	A	B	R	A	C
4	3	7					A	D	A	C	R		
5	0	5						A	D	A	C	R	

```
public static int search(String pat, String txt)
{
   int M = pat.length();
   int N = txt.length();
   for (int i = 0; i <= N - M; i++)
   {
      int j;
      for (j = 0; j < M; j++)
         if (txt.charAt(i+j) != pat.charAt(j))
            break;
      if (j == M) return i;   ← index in text where pattern starts
   }
   return N;   ← not found
}
```


Brute-force substring search: worst case

Brute-force algorithm can be slow if text and pattern are repetitive.

i	j	i+j	0	1	2	3	4	5	6	7	8	9
		txt →	A	A	A	A	A	A	A	A	A	B
0	4	4	A	A	A	A	B ← pat					
1	4	5		A	A	A	A	B				
2	4	6			A	A	A	A	B			
3	4	7				A	A	A	A	B		
4	4	8					A	A	A	A	B	
5	5	10						A	A	A	A	B

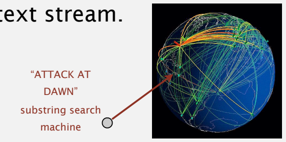
match

Worst case. $\sim M N$ char compares.

Backup

In many applications, we want to avoid backup in text stream.

- Treat input as stream of data.
- Abstract model: standard input.



Brute-force algorithm needs backup for every mismatch.

matched chars

mismatch

```
A A A A A A A A A A A A A A A A A A A A A A B
                  A A A A A B
```

backup

```
A A A A A A A A A A A A A A A A A A A A A A B
                    A A A A A B
```

shift pattern right one position

Approach 1. Maintain buffer of last M characters.

Approach 2. Stay tuned.

Brute-force substring search: alternate implementation

Same sequence of char compares as previous implementation.

- `i` points to end of sequence of already-matched chars in text.
- `j` stores # of already-matched chars (end of sequence in pattern).

i	j	0	1	2	3	4	5	6	7	8	9	10
		A	B	A	C	A	D	A	B	R	A	C
7	3					A	D	A	C	R		
5	0						A	D	A	C	R	

```
public static int search(String pat, String txt)
{
   int i, N = txt.length();
   int j, M = pat.length();
   for (i = 0, j = 0; i < N && j < M; i++)
   {
      if (txt.charAt(i) == pat.charAt(j)) j++;
      else { i -= j; j = 0;  }
   }
   if (j == M) return i - M;
   else        return N;
}
```

explicit backup

Algorithmic challenges in substring search

Brute-force is not always good enough.

Theoretical challenge. Linear-time guarantee. ← fundamental algorithmic problem

Practical challenge. Avoid backup in text stream. ← often no room or time to save text

```
Now is the time for all people to come to the aid of their party. Now is the time for all good people to
come to the aid of their party. Now is the time for many good people to come to the aid of their party.
Now is the time for all good people to come to the aid of their party. Now is the time for a lot of good
people to come to the aid of their party. Now is the time for all of the good people to come to the aid of
their party. Now is the time for all good people to come to the aid of their party. Now is the time for
each good person to come to the aid of their party. Now is the time for all good people to come to the aid
of their party. Now is the time for all good Republicans to come to the aid of their party. Now is the
time for all good people to come to the aid of their party. Now is the time for many or all good people to
come to the aid of their party. Now is the time for all good people to come to the aid of their party. Now
is the time for all good Democrats to come to the aid of their party. Now is the time for all people to
come to the aid of their party. Now is the time for all good people to come to the aid of their party. Now
is the time for many good people to come to the aid of their party. Now is the time for all good people to
come to the aid of their party. Now is the time for a lot of good people to come to the aid of their
party. Now is the time for all of the good people to come to the aid of their party. Now is the time for
all good people to come to the aid of their attack at dawn party. Now is the time for each person to come
to the aid of their party. Now is the time for all good people to come to the aid of their party. Now is
the time for all good Republicans to come to the aid of their party. Now is the time for all good people
to come to the aid of their party. Now is the time for many or all good people to come to the aid of their
party. Now is the time for all good people to come to the aid of their party. Now is the time for all good
Democrats to come to the aid of their party.
```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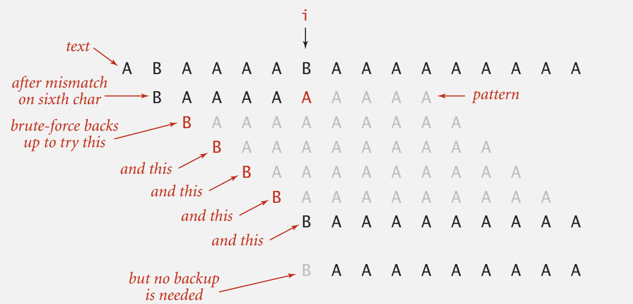
5.3 Substring Search

Knuth-Morris-Pratt substring search

Intuition. Suppose we are searching in text for pattern BAAAAAAAAA.

- Suppose we match 5 chars in pattern, with mismatch on 6^{th} char.
- We know previous 6 chars in text are BAAAAB. ← assuming { A, B } alphabet
- Don't need to back up text pointer!



Knuth-Morris-Pratt algorithm. Clever method to always avoid backup. (!)

Deterministic finite state automaton (DFA)

DFA is abstract string-searching machine.

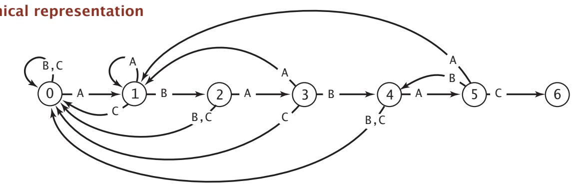
- Finite number of states (including start and halt).
- Exactly one transition for each char in alphabet.
- Accept if sequence of transitions leads to halt state.

internal representation

	j	0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
dfa[][j]	A	1	1	3	1	5	1
	B	0	2	0	4	0	4
	C	0	0	0	0	0	6

If in state j reading char c:
if j is 6 halt and accept
else move to state dfa[c][j]

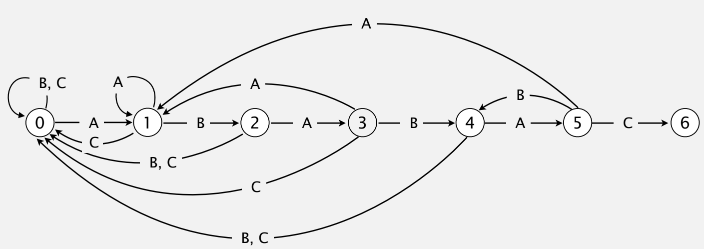
graphical representation



DFA simulation demo

A A B A C A A B A B A C A A

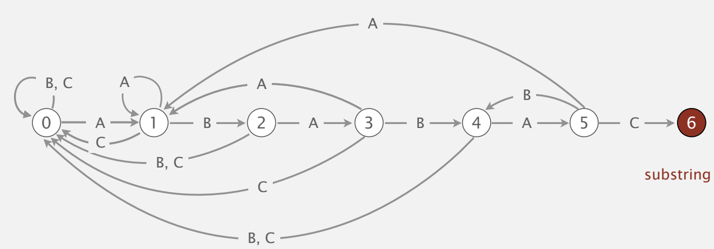
		0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
dfa[][j]	A	1	1	3	1	5	1
	B	0	2	0	4	0	4
	C	0	0	0	0	0	6



DFA simulation demo

A A B A C A A B A B A C A A

		0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
dfa[][j]	A	1	1	3	1	5	1
	B	0	2	0	4	0	4
	C	0	0	0	0	0	6



Interpretation of Knuth-Morris-Pratt DFA

Q. What is interpretation of DFA state after reading in `txt[i]`?

A. State = number of characters in pattern that have been matched.

length of longest prefix of `pat[]` that is a suffix of `txt[0..i]`

Ex. DFA is in state 3 after reading in `txt[0..6]`.

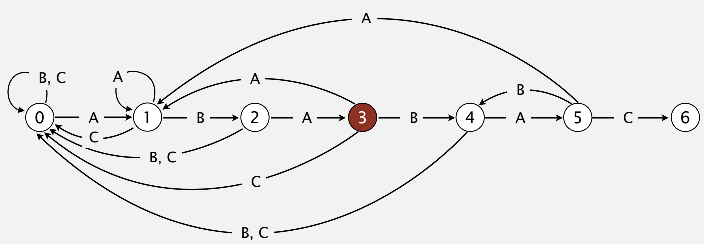
txt (i = 6)

	0	1	2	3	4	5	6	7	8
txt	B	C	B	A	A	B	A	C	A

suffix of `txt[0..6]`

	0	1	2	3	4	5
pat	A	B	A	B	A	C

prefix of `pat[]`



Knuth-Morris-Pratt substring search: Java implementation

Key differences from brute-force implementation.

- Need to precompute `dfa[][]` from pattern.
- Text pointer `i` never decrements.

```
public int search(String txt)
{
   int i, j, N = txt.length();
   for (i = 0, j = 0; i < N && j < M; i++)
      j = dfa[txt.charAt(i)][j];
   if (j == M) return i - M;
   else        return N;
}
```

no backup

Running time.

- Simulate DFA on text: at most N character accesses.
- Build DFA: how to do efficiently? [warning: tricky algorithm ahead]

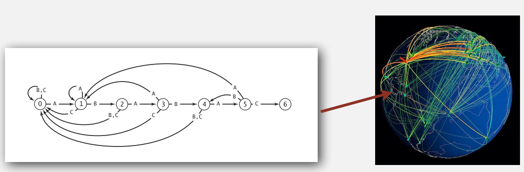
Knuth-Morris-Pratt substring search: Java implementation

Key differences from brute-force implementation.

- Need to precompute `dfa[][]` from pattern.
- Text pointer `i` never decrements.
- Could use input stream.

```
public int search(In in)
{
   int i, j;
   for (i = 0, j = 0; !in.isEmpty() && j < M; i++)
      j = dfa[in.readChar()][j];
   if (j == M) return i - M;
   else        return NOT_FOUND;
}
```

no backup



Knuth-Morris-Pratt construction demo

Include one state for each character in pattern (plus accept state).

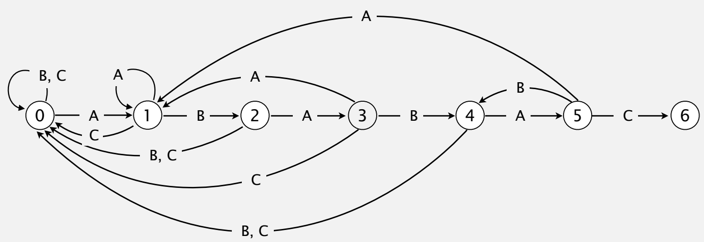
		0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
dfa[][j]	A						
	B						
	C						

Constructing the DFA for KMP substring search for A B A B A C

Knuth-Morris-Pratt construction demo

		0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
dfa[][j]	A	1	1	3	1	5	1
	B	0	2	0	4	0	4
	C	0	0	0	0	0	6

Constructing the DFA for KMP substring search for A B A B A C



How to build DFA from pattern?

Include one state for each character in pattern (plus accept state).

		0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
dfa[][j]	A						
	B						
	C						

How to build DFA from pattern?

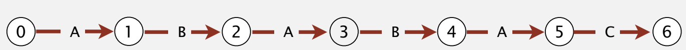
Match transition. If in state `j` and next char `c == pat.charAt(j)`, go to `j+1`.

first j characters of pattern have already been matched

next char matches

now first j+1 characters of pattern have been matched

		0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
dfa[][j]	A	1		3		5	
	B		2		4		
	C						6



How to build DFA from pattern?

Mismatch transition. If in state `j` and next char `c != pat.charAt(j)`, then the last `j-1` characters of input are `pat[1..j-1]`, followed by `c`.

To compute `dfa[c][j]`: Simulate `pat[1..j-1]` on DFA and take transition `c`.
Running time. Seems to require j steps.

still under construction (!)

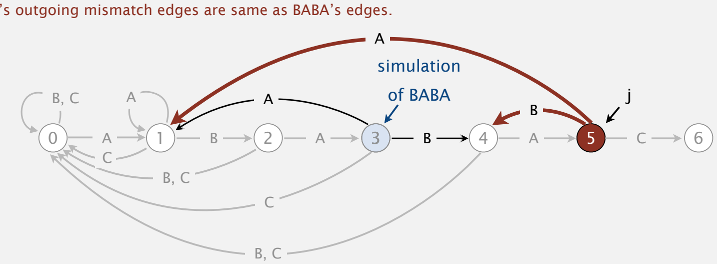
Ex. `dfa['A'][5] = 1;` `dfa['B'][5] = 4`

simulate BABA; take transition 'A' = dfa['A'][3]

simulate BABA; take transition 'B' = dfa['B'][3]

5's outgoing mismatch edges are same as BABA's edges.

j	0	1	2	3	4	5
pat.charAt(j)	A	B	A	B	A	C



How to build DFA from pattern?

Mismatch transition. If in state `j` and next char `c != pat.charAt(j)`, then the last `j-1` characters of input are `pat[1..j-1]`, followed by `c`.

state X

To compute `dfa[c][j]`: Simulate `pat[1..j-1]` on DFA and take transition `c`.
Running time. Takes only constant time if we maintain state X.

Update X as well!

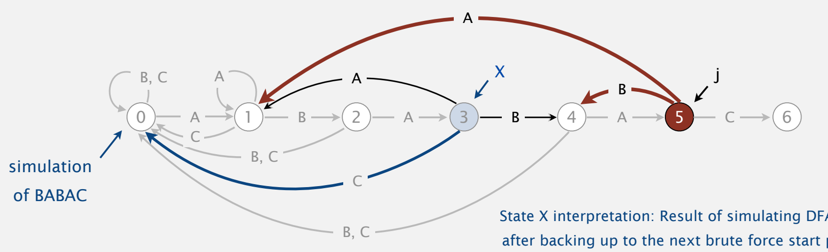
Ex. `dfa['A'][5] = 1;` `dfa['B'][5] = 4` `X -> 0`

from state X, take transition 'A' = dfa['A'][X] = 1

from state X, take transition 'B' = dfa['B'][X] = 4

from state X, take transition 'C' X = dfa['C'][X] = 0

0	1	2	3	4	5
A	B	A	B	A	C



State X interpretation: Result of simulating DFA on text after backing up to the next brute force start position.

Knuth-Morris-Pratt construction demo (in linear time)

Include one state for each character in pattern (plus accept state).

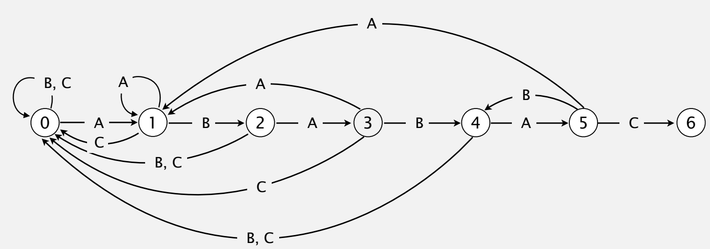
		0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
dfa[][j]	A						
	B						
	C						

Constructing the DFA for KMP substring search for A B A B A C

Knuth-Morris-Pratt construction demo (in linear time)

		0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
dfa[][j]	A	1	1	3	1	5	1
	B	0	2	0	4	0	4
	C	0	0	0	0	0	6

Constructing the DFA for KMP substring search for A B A B A C



Constructing the DFA for KMP substring search: Java implementation

For each state `j`:

- Copy `dfa[][X]` to `dfa[][j]` for mismatch case.
- Set `dfa[pat.charAt(j)][j]` to `j+1` for match case.
- Update `X`.

```
public KMP(String pat)
{
   this.pat = pat;
   M = pat.length();
   dfa = new int[R][M];
   dfa[pat.charAt(0)][0] = 1;
   for (int X = 0, j = 1; j < M; j++)
   {
      for (int c = 0; c < R; c++)
         dfa[c][j] = dfa[c][X];          // copy mismatch cases
      dfa[pat.charAt(j)][j] = j+1;       // set match case
      X = dfa[pat.charAt(j)][X];         // update restart state
   }
}
```

Running time. M character accesses (but space/time proportional to $R M$).

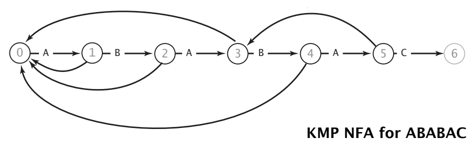
KMP substring search analysis

Proposition. KMP substring search accesses no more than $M + N$ chars to search for a pattern of length M in a text of length N.

Pf. Each pattern char accessed once when constructing the DFA; each text char accessed once (in the worst case) when simulating the DFA.

Proposition. KMP constructs `dfa[][]` in time and space proportional to $R\,M$.

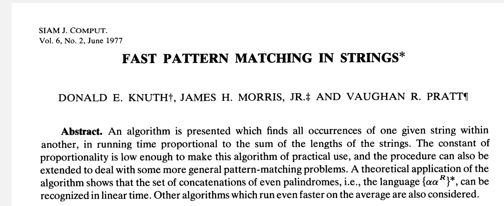
Larger alphabets. Improved version of KMP constructs `nfa[]` in time and space proportional to M.



KMP NFA for ABABAC

Knuth-Morris-Pratt: brief history

- Independently discovered by two theoreticians and a hacker.
  - Knuth: inspired by esoteric theorem, discovered linear algorithm
  - Pratt: made running time independent of alphabet size
  - Morris: built a text editor for the CDC 6400 computer
- Theory meets practice.


SIAM J. COMPUT.
Vol. 6, No. 2, June 1977

FAST PATTERN MATCHING IN STRINGS*

DONALD E. KNUTH†, JAMES H. MORRIS, JR.‡ AND VAUGHAN R. PRATT¶

Abstract. An algorithm is presented which finds all occurrences of one given string within another, in running time proportional to the sum of the lengths of the strings. The constant of proportionality is low enough to make this algorithm of practical use, and the procedure can also be extended to deal with some more general pattern-matching problems. A theoretical application of the algorithm shows that the set of concatenations of even palindromes, i.e., the language $\{\alpha\alpha^R\}^*$, can be recognized in linear time. Other algorithms which run even faster on the average are also considered.


Don Knuth


Jim Morris


Vaughan Pratt

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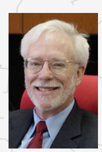
Robert Boyer


J. Strother Moore

Boyer-Moore: mismatched character heuristic

Intuition.

- Scan characters in pattern from right to left.
- Can skip as many as M text chars when finding one not in the pattern.

overall text scanning still goes left to right!

i	j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
text →		F	I	N	D	I	N	A	H	A	Y	S	T	A	C	K	N	E	E	D	L	E	I	N	A
0	5	N	E	E	D	L	E ← pattern																		
5	5						N	E	E	D	L	E													
11	4												N	E	E	D	L	E							
15	0																N	E	E	D	L	E			

return i = 15

Boyer-Moore: mismatched character heuristic

Q. How much to skip?

i.e. How much do we move i to the right?

Case 1. Mismatch character not in pattern.

before

```
                i
                ↓
txt  . . . . . . T L E . . . . . .
pat             N E E D L E
```

after

```
                    i
                    ↓
txt  . . . . . . T L E . . . . . .
pat                 N E E D L E
```

mismatch character 'T' not in pattern: increment i one character beyond 'T'

Boyer-Moore: mismatched character heuristic

Q. How much to skip?

Case 2a. Mismatch character in pattern.

before

```
                i
                ↓
txt  . . . . . . N L E . . . . . .
pat             N E N D L E
```

after

```
                  i
                  ↓
txt  . . . . . . N L E . . . . . .
pat               N E N D L E
```

mismatch character 'N' in pattern: align text 'N' with rightmost pattern 'N'

Observation: L will immediately mismatch E! KMP-like version of Boyer-Moore avoids this.

Boyer-Moore: mismatched character heuristic

Q. How much to skip?

Case 2b. Mismatch character in pattern (but heuristic no help).

before

```
                i
                ↓
txt  . . . . . . E L E . . . . . .
pat             N E E D L E
```

i going backwards is bad!

aligned with rightmost E?

```
            i
            ↓
txt  . . . . . . E L E . . . . . .
pat         N E E D L E
```

mismatch character 'E' in pattern: align text 'E' with rightmost pattern 'E' ?

Boyer-Moore: mismatched character heuristic

Q. How much to skip?

Case 2b. Mismatch character in pattern (but heuristic no help).

before

```
                i
                ↓
txt  . . . . . . E L E . . . . . .
pat             N E E D L E
```

after

```
                  i
                  ↓
txt  . . . . . . E L E . . . . . .
pat               N E E D L E
```

mismatch character 'E' in pattern: increment i by 1

Boyer-Moore: mismatched character heuristic

Q. How much to skip?

A. Precompute index of rightmost occurrence of character **c** in pattern (-1 if character not in pattern).

```
right = new int[R];
for (int c = 0; c < R; c++)
   right[c] = -1;
for (int j = 0; j < M; j++)
   right[pat.charAt(j)] = j;
```

		N	E	E	D	L	E	
c		0	1	2	3	4	5	right[c]
A	-1	-1	-1	-1	-1	-1	-1	-1
B	-1	-1	-1	-1	-1	-1	-1	-1
C	-1	-1	-1	-1	-1	-1	-1	-1
D	-1	-1	-1	-1	3	3	3	3
E	-1	-1	1	2	2	2	5	5
...								-1
L	-1	-1	-1	-1	-1	4	4	4
M	-1	-1	-1	-1	-1	-1	-1	-1
N	-1	0	0	0	0	0	0	0
...								-1

Boyer-Moore skip table computation

Boyer-Moore: Java implementation

```
public int search(String txt)
{
   int N = txt.length();
   int M = pat.length();
   int skip;
   for (int i = 0; i <= N-M; i += skip)
   {
      skip = 0;
      for (int j = M-1; j >= 0; j--)
      {
         if (pat.charAt(j) != txt.charAt(i+j))
         {
            skip = Math.max(1, j - right[txt.charAt(i+j)]);
            break;
         }
      }
      if (skip == 0) return i;
   }
   return N;
}
```

compute skip value

in case other term is nonpositive

match

Boyer-Moore: analysis

Property. Substring search with the Boyer-Moore mismatched character heuristic takes about $\sim N / M$ character compares to search for a pattern of length M in a text of length N.

sublinear!

Worst-case. Can be as bad as $\sim M N$.

i	skip		0	1	2	3	4	5	6	7	8	9	
		txt →	B	B	B	B	B	B	B	B	B	B	
0	0		A	B	B	B	B						← pat
1	1			A	B	B	B	B					
2	1				A	B	B	B	B				
3	1					A	B	B	B	B			
4	1						A	B	B	B	B		
5	1							A	B	B	B	B	

Boyer-Moore variant. Can improve worst case to $\sim 3\,N$ character compares by adding a KMP-like rule to guard against repetitive patterns.

http://algs4.cs.princeton.edu

5.3 Substring Search

Michael Rabin, Turing Award '76

Dick Karp, Turing Award '85

Rabin-Karp fingerprint search

Basic idea = modular hashing.

- Compute a hash of pattern characters 0 to $M - 1$.
- For each i, compute a hash of text characters i to $M + i - 1$.
- If pattern hash = text substring hash, check for a match.

```
      pat.charAt(i)
i     0  1  2  3  4
      2  6  5  3  5   % 997 = 613

                          txt.charAt(i)
i     0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15
      3  1  4  1  5  9  2  6  5  3  5  8  9  7  9  3
0     3  1  4  1  5  % 997 = 508
1        1  4  1  5  9  % 997 = 201
2           4  1  5  9  2  % 997 = 715
3              1  5  9  2  6  % 997 = 971
4                 5  9  2  6  5  % 997 = 442
5                    9  2  6  5  3  % 997 = 929   match
6  ← return i = 6       2  6  5  3  5  % 997 = 613
```

Goal. Find efficient way to compute hashes for each position in text.

Review: Computing hash functions for a string (from hashing lecture)

Modular hash function. Using the notation t_i for `txt.charAt(i)`, we wish to compute

$$x_i = t_i R^{M-1} + t_{i+1} R^{M-2} + \ldots + t_{i+M-1} R^{0} \pmod{Q}$$

Intuition. M-digit, base-R integer, modulo Q.

Horner's method. Linear-time method to evaluate degree-M polynomial.

```
      pat.charAt()
i     0  1  2  3  4
      2  6  5  3  5
0     2  % 997 = 2                  R              Q
1     2  6  % 997 = (2*10 + 6) % 997 = 26
2     2  6  5  % 997 = (26*10 + 5) % 997 = 265
3     2  6  5  3  % 997 = (265*10 + 3) % 997 = 659
4     2  6  5  3  5  % 997 = (659*10 + 5) % 997 = 613
```

```
// Compute hash for M-digit key
private long hash(String key, int M)
{
   long h = 0;
   for (int j = 0; j < M; j++)
      h = (h * R + key.charAt(j)) % Q;
   return h;
}
```

Slow way to compute hashCode: **"26535" = 2*10000 + 6*1000 + 5*100 + 3*10 + 5**

Horner's method to compute hashCode: **"26535" = ((((2) *10 + 6) * 10 + 5) * 10 + 3) * 10 + 5**

R = 10

Modulus trick

Modulus property. Given an expression % Q, if the expression contains only multiplication and addition operations, can replace any term X by X % Q.

Examples.

```
    (2653 * 10 + 5) % 997
 =  ( 659 * 10 + 5) % 997
```

```
   ((48500 - 4*10000) * 10 + 6) % 9
=  ((8     - 4*1    ) *  1 + 6) % 9
=  ((8     - 4*1    ) * 10 + 6) % 9
```

Don't have to replace all of them!

Efficiently computing the hash function (incremental approach)

Challenge. How to efficiently compute x_{i+1} given that we know x_i.

$$x_i = t_i R^{M-1} + t_{i+1} R^{M-2} + \ldots + t_{i+M-1} R^0$$

$$x_{i+1} = t_{i+1} R^{M-1} + t_{i+2} R^{M-2} + \ldots + t_{i+M} R^0$$

Key property. Can update hash function in constant time!

$$x_{i+1} = (\underbrace{x_i}_{\text{current value}} - \underbrace{t_i R^{M-1}}_{\text{subtract leading digit}})\ \underbrace{R}_{\text{multiply by radix}} + \underbrace{t_{i+M}}_{\text{add new trailing digit}}$$

combine with mod trick to get hash function

(can precompute R^{M-1})

```
i              ...  2  3  4  5  6  7  ...
current value   1   4  1  5  9  2  6  5    text
    new value       4  1  5  9  2  6  5

                    4  1  5  9  2    current value
                 -  4  0  0  0  0
                       1  5  9  2    subtract leading digit
                          *  1  0    multiply by radix
                    1  5  9  2  0
                             +  6    add new trailing digit
                    1  5  9  2  6    new value
```

Example: Using mod trick and incremental approach to get hash

Given

```
48500 % 9 = 8
```

x_i

Find

```
85006 % 9
```

x_{i+1}

Hint

```
10000 % 9 = 1
```

R^{M-1}, also called RM

Solution

t_i R t_{i+M-1} Q

```
85006     = (48500  - 4*10000) * 10 + 6
85006 % 9 = ((48500 - 4*10000) * 10 + 6) % 9
          = ((8     - 4*1    ) * 1  + 6) % 9
          = 10 % 9
          = 1
```

Key idea: To compute x_{i+1}, use x_i, t_i, t_{i+M-1}, RM, R, and Q.

Rabin-Karp substring search example

First 5 entries. Horner's rule.

Later entries. Use incremental approach and mod trick.

```
i    0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15
     3  1  4  1  5  9  2  6  5  3  5  8  9  7  9  3
0    3  % 997 = 3
1    3  1  % 997 = (3*10 + 1) % 997 = 31
2    3  1  4  % 997 = (31*10 + 4) % 997 = 314
3    3  1  4  1  % 997 = (314*10 + 1) % 997 = 150
4    3  1  4  1  5  % 997 = (150*10 + 5) % 997 = 508
5       1  4  1  5  9  % 997 = ((508 + 3*(997 - 30))*10 + 9) % 997 = 201
6          4  1  5  9  2  % 997 = ((201 + 1*(997 - 30))*10 + 2) % 997 = 715
7             1  5  9  2  6  % 997 = ((715 + 4*(997 - 30))*10 + 6) % 997 = 971
8                5  9  2  6  5  % 997 = ((971 + 1*(997 - 30))*10 + 5) % 997 = 442
9                   9  2  6  5  3  % 997 = ((442 + 5*(997 - 30))*10 + 3) % 997 = 929
10 ← return i-M+1 = 6  2  6  5  3  5  % 997 = ((929 + 9*(997 - 30))*10 + 5) % 997 = 613
```

Q RM R *match*

This extra factor of Q is added to ensure that the sum is positive since % is the remainder operator, not the modulus operator in Java.

Rabin-Karp: Java implementation

```
public class RabinKarp
{
   private long patHash;    // pattern hash value
   private int M;           // pattern length
   private long Q;          // modulus
   private int R;           // radix
   private long RM;         // R^(M-1) % Q

   public RabinKarp(String pat) {
      M = pat.length();
      R = 256;
      Q = longRandomPrime();

      RM = 1;
      for (int i = 1; i <= M-1; i++)
         RM = (R * RM) % Q;
      patHash = hash(pat, M);
   }

   private long hash(String key, int M)
   {  /* as before */  }

   public int search(String txt)
   {  /* see next slide */  }
}
```

a large prime (but avoid overflow)

precompute R^{M-1} (mod Q) equivalent of 10000 % 9 from example

Rabin-Karp: Java implementation (continued)

Monte Carlo version. Return match if hash match.

```
public int search(String txt)
{
   int N = txt.length();
   int txtHash = hash(txt, M);
   if (patHash == txtHash) return 0;
   for (int i = M; i < N; i++)
   {
      txtHash = (txtHash + Q - RM*txt.charAt(i-M) % Q) % Q;
      txtHash = (txtHash*R + txt.charAt(i)) % Q;
      if (patHash == txtHash) return i - M + 1;
   }
   return N;
}
```

check for hash collision using rolling hash function

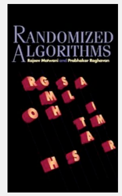
ensures new txthash is positive, doesn't affect magnitude of answer

Las Vegas version. Check for substring match if hash match; continue search if false collision.

Rabin-Karp analysis

Theory. If Q is a sufficiently large random prime (about $M N^2$), then the probability of a false collision is about $1 / N$.

Practice. Choose Q to be a large prime (but not so large to cause overflow). Under reasonable assumptions, probability of a collision is about $1 / Q$.



Monte Carlo version.

- Always runs in linear time.
- Extremely likely to return correct answer (but not always!).

Las Vegas version.

- Always returns correct answer.
- Extremely likely to run in linear time (but worst case is $M N$).

Rabin-Karp fingerprint search

Advantages.

- Extends to 2d patterns.
- Extends to finding multiple patterns.

Disadvantages.

- Arithmetic ops slower than char compares.
- Las Vegas version requires backup.
- Poor worst-case guarantee.

Q. How would you extend Rabin-Karp to efficiently search for any one of P possible patterns in a text of length N?

Substring search cost summary

Cost of searching for an M-character pattern in an N-character text.

algorithm	version	operation count: guarantee	operation count: typical	backup in input?	correct?	extra space
brute force	—	$M N$	$1.1\ N$	*yes*	*yes*	1
Knuth-Morris-Pratt	*full DFA (Algorithm 5.6)*	$2\ N$	$1.1\ N$	*no*	*yes*	MR
	mismatch transitions only	$3\ N$	$1.1\ N$	*no*	*yes*	M
Boyer-Moore	*full algorithm*	$3\ N$	N / M	*yes*	*yes*	R
	mismatched char heuristic only (Algorithm 5.7)	$M N$	N / M	*yes*	*yes*	R
Rabin-Karp†	*Monte Carlo (Algorithm 5.8)*	$7\ N$	$7\ N$	*no*	*yes†*	1
	Las Vegas	$7\ N$†	$7\ N$	*yes*	*yes*	1

† probabilisitic guarantee, with uniform hash function