Algorithms

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4.4 Shortest Paths

- *APIs*
- *shortest-paths properties*
- *Dijkstra's algorithm*
- *edge-weighted DAGs*
- *negative weights*

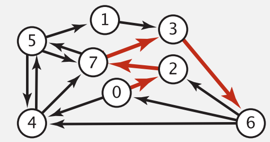
Shortest paths in an edge-weighted digraph

Given an edge-weighted digraph, find the shortest path from s to t.

Related to, but not the same as the MST problem

edge-weighted digraph

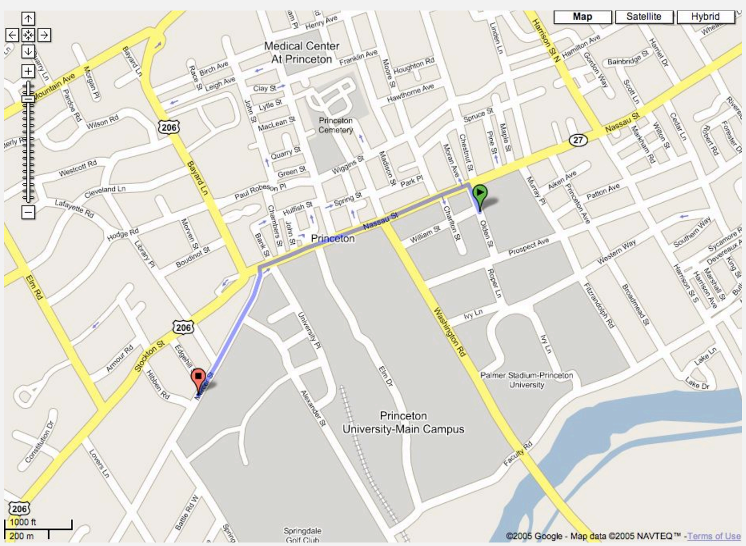
4->5	0.35
5->4	0.35
4->7	0.37
5->7	0.28
7->5	0.28
5->1	0.32
0->4	0.38
0->2	0.26
7->3	0.39
1->3	0.29
2->7	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93



shortest path from 0 to 6

0->2	0.26
2->7	0.34
7->3	0.39
3->6	0.52

Google maps



Car navigation

Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.

http://en.wikipedia.org/wiki/Seam_carving

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

Shortest path variants

Which vertices?

- Single source: from one vertex s to every other vertex.
- Single sink: from every vertex to one vertex t.
- Source-sink: from one vertex s to another t.
- All pairs: between all pairs of vertices.

Restrictions on edge weights?

- Nonnegative weights.
- Euclidean weights.
- Arbitrary weights.

Cycles?

- No directed cycles.
- No "negative cycles."

Simplifying assumption. Shortest paths from s to each vertex v exist.

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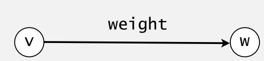
- *negative weights*

Weighted directed edge API

public class DirectedEdge

	`DirectedEdge(int v, int w, double weight)`	*weighted edge v→w*
`int`	`from()`	*vertex v*
`int`	`to()`	*vertex w*
`double`	`weight()`	*weight of this edge*
`String`	`toString()`	*string representation*



Idiom for processing an edge `e`: `int v = e.from(), w = e.to();`

Weighted directed edge: implementation in Java

Similar to `Edge` for undirected graphs, but a bit simpler.

```
public class DirectedEdge
{
   private final int v, w;
   private final double weight;

   public DirectedEdge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int from()
   {  return v;  }

   public int to()
   {  return w;  }

   public int weight()
   {  return weight; }
}
```

from() and to() replace either() and other()

Edge-weighted digraph API

public class	EdgeWeightedDigraph	
	EdgeWeightedDigraph(int V)	*edge-weighted digraph with V vertices*
	EdgeWeightedDigraph(In in)	*edge-weighted digraph from input stream*
void	addEdge(DirectedEdge e)	*add weighted directed edge e*
Iterable<DirectedEdge>	adj(int v)	*edges pointing from v*
int	V()	*number of vertices*
int	E()	*number of edges*
Iterable<DirectedEdge>	edges()	*all edges*
String	toString()	*string representation*

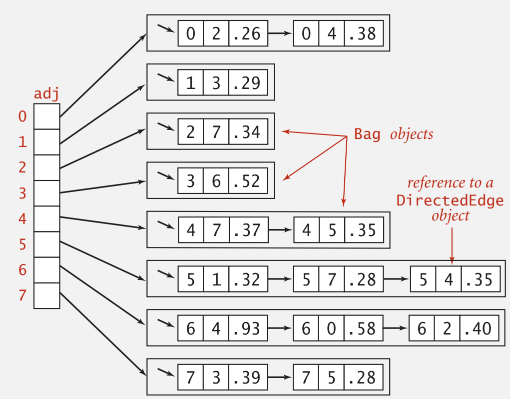
Conventions. Allow self-loops and parallel edges.

Edge-weighted digraph: adjacency-lists representation

tinyEWD.txt

V → 8
E → 15

```
8
15
4 5  0.35
5 4  0.35
4 7  0.37
5 7  0.28
7 5  0.28
5 1  0.32
0 4  0.38
0 2  0.26
7 3  0.39
1 3  0.29
2 7  0.34
6 2  0.40
3 6  0.52
6 0  0.58
6 4  0.93
```



Edge-weighted digraph: adjacency-lists implementation in Java

Same as `EdgeWeightedGraph` except replace `Graph` with `Digraph`.

```
public class EdgeWeightedDigraph
{
   private final int V;
   private final Bag<DirectedEdge>[] adj;

   public EdgeWeightedDigraph(int V)
   {
      this.V = V;
      adj = (Bag<DirectedEdge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<DirectedEdge>();
   }

   public void addEdge(DirectedEdge e)
   {
      int v = e.from();
      adj[v].add(e);
   }

   public Iterable<DirectedEdge> adj(int v)
   {  return adj[v];  }
}
```

add edge e = v→w to only v's adjacency list

Single-source shortest paths API

Goal. Find the shortest path from s to every other vertex.

public class SP

	SP(EdgeWeightedDigraph G, int s)	*shortest paths from s in graph G*
double	distTo(int v)	*length of shortest path from s to v*
Iterable <DirectedEdge>	pathTo(int v)	*shortest path from s to v*
boolean	hasPathTo(int v)	*is there a path from s to v?*

```
SP sp = new SP(G, s);
for (int v = 0; v < G.V(); v++)
{
   StdOut.printf("%d to %d (%.2f): ", s, v, sp.distTo(v));
   for (DirectedEdge e : sp.pathTo(v))
      StdOut.print(e + "  ");
   StdOut.println();
}
```


Single-source shortest paths API

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double	distTo(int v)	*length of shortest path from s to v*
Iterable <DirectedEdge>	pathTo(int v)	*shortest path from s to v*
boolean	hasPathTo(int v)	*is there a path from s to v?*

```
% java SP tinyEWD.txt 0
0 to 0 (0.00):
0 to 1 (1.05): 0->4 0.38  4->5 0.35  5->1 0.32
0 to 2 (0.26): 0->2 0.26
0 to 3 (0.99): 0->2 0.26  2->7 0.34  7->3 0.39
0 to 4 (0.38): 0->4 0.38
0 to 5 (0.73): 0->4 0.38  4->5 0.35
0 to 6 (1.51): 0->2 0.26  2->7 0.34  7->3 0.39  3->6 0.52
0 to 7 (0.60): 0->2 0.26  2->7 0.34
```


4.4 Shortest Paths

- APIs
- *shortest-paths properties*
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights

Data structures for single-source shortest paths

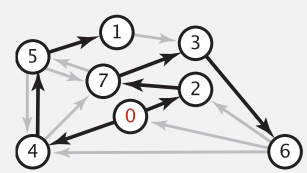
Compare to MST problem

Goal. Find the shortest path from s to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- distTo[v] is length of shortest path from s to v.
- edgeTo[v] is last edge on shortest path from s to v.



shortest-paths tree from 0

	edgeTo[]	distTo[]
0	null	0
1	5->1 0.32	1.05
2	0->2 0.26	0.26
3	7->3 0.37	0.97
4	0->4 0.38	0.38
5	4->5 0.35	0.73
6	3->6 0.52	1.49
7	2->7 0.34	0.60

parent-link representation

Data structures for single-source shortest paths

Goal. Find the shortest path from s to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- `distTo[v]` is length of shortest path from s to v.
- `edgeTo[v]` is last edge on shortest path from s to v.

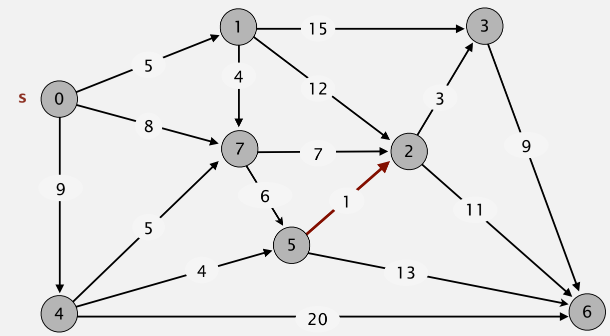
```
public double distTo(int v)
{  return distTo[v];  }
```

```
public Iterable<DirectedEdge> pathTo(int v)
{
   Stack<DirectedEdge> path = new Stack<DirectedEdge>();
   for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
      path.push(e);
   return path;
}
```


Kruskal's Algorithm on Directed Graphs

Starting from a list containing all edges sorted in ascending weight order.

- Iterate through list in ascending order. Add to the SPT unless this creates a cycle.



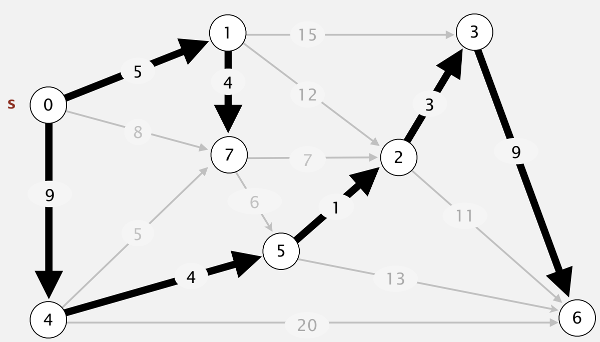
v	distTo[]	edgeTo[]
0	0.0	-
1	∞	-
2	∞	-
3	∞	-
4	∞	-
5	∞	-
6	∞	-
7	∞	-

Q: Is this algorithm correct?
A. No
B. Yes

Lazy Prim's Algorithm on Directed Graphs

Starting from a list containing all edges sorted in ascending weight order.

- Iterate through list in ascending order. Add to the SPT unless the target vertex is already marked.



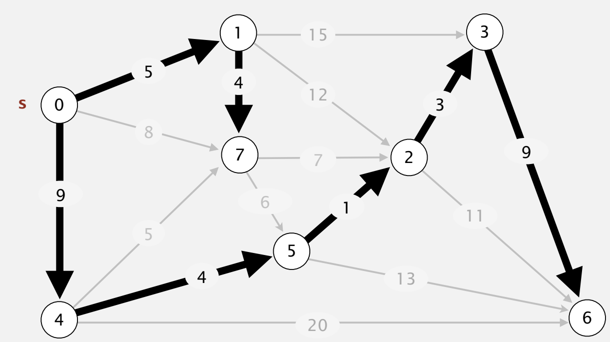
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0
2	14.0	5
3	17.0	2
4	9.0	0
5	13.0	4
6	26.0	3
7	9.0	1

Q: Is this algorithm correct?
A. No
B. Yes

Lazy Prim's Algorithm on Directed Graphs

Starting from a list containing all edges sorted in ascending weight order.

- Iterate through list in ascending order. Add to the SPT unless the target vertex is already marked.



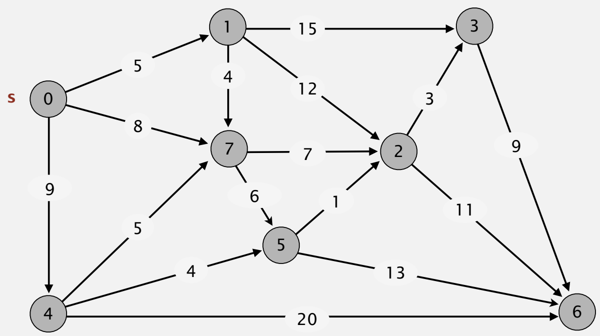
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0
2	14.0	5
3	17.0	2
4	9.0	0
5	13.0	4
6	26.0	3
7	9.0	1

Observation for e = 0→7 Easy shortcut!
`e.weight = 8.0`
`distTo[7] = 9.0`

Lazy Prim's Algorithm on Directed Graphs

Starting with a priority queue containing s's outgoing edges.

- Remove min edge from PQ. Add to the SPT unless this creates a cycle.
- Enqueue any discovered edges.



v	distTo[]	edgeTo[]
0	0.0	-
1	∞	-
2	∞	-
3	∞	-
4	∞	-
5	∞	-
6	∞	-
7	∞	-

Q: Is this algorithm correct?
A. No
B. Yes

Lazy Prim's Algorithm on Directed Graphs

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- Remove min edge from PQ. Add to the SPT unless this creates a cycle.
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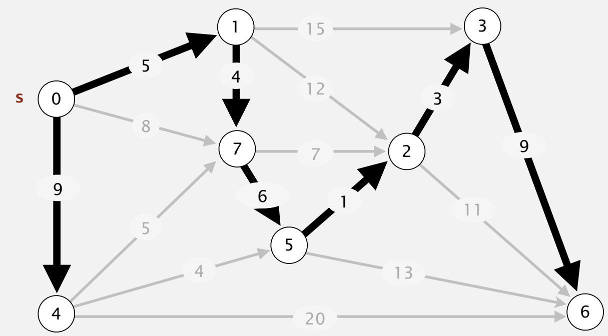


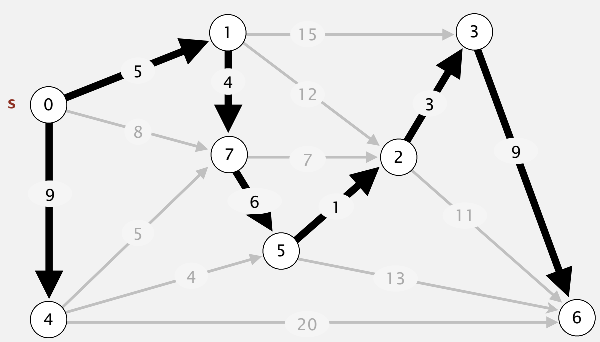
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0
2	16.0	5
3	19.0	3
4	9.0	0
5	15.0	7
6	28.0	3
7	9.0	1

Q: Is this algorithm correct?
A. No
B. Yes

Lazy Prim's Algorithm on Directed Graphs

Starting with a priority queue containing s's outgoing edges.

- Remove min edge from PQ. Add to the SPT unless this creates a cycle.
- Enqueue any discovered edges.



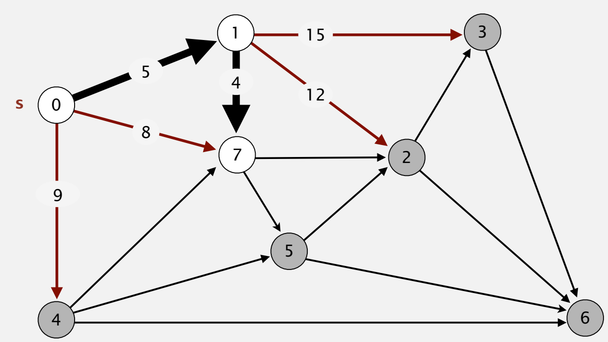
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0
2	16.0	5
3	19.0	3
4	9.0	0
5	15.0	7
6	28.0	3
7	9.0	1

Observation for e = 0→7 Same easy shortcut!
e.weight = 8.0
distTo[7] = 9.0

Lazy Prim's Algorithm on Directed Graphs

Fundamental distinction between MST and SPT

- SPT: What matters is the distance from the **source**, not the distance to the **tree**!



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0
2	∞	-
3	∞	-
4	9.0	0
5	∞	-
6	∞	-
7	∞	-

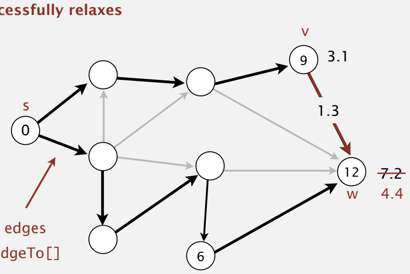
- Non-obvious fact: We'd like a way to deal with incorrect choices.
  - Want some way to allow 0-7 to take over from 1-7.

Edge relaxation (i.e. examine edge and use if better)

Relax edge $e = v \to w$.

- `distTo[v]` is length of shortest known path from s to v.
- `distTo[w]` is length of shortest known path from s to w.
- `edgeTo[w]` is last edge on shortest known path from s to w.
- If e = v→w gives shorter path to w through v, update both `distTo[w]` and `edgeTo[w]`.

v→w successfully relaxes



black edges are in edgeTo[]

but maybe not shortest

Table of known paths

v#	distTo[]	edgeTo[]
...		
9	3.1	[omitted]
12	~~7.2~~ 4.4	~~6~~ 9

Edge relaxation (i.e. examine edge and use if better)

Relax edge $e = v \to w$.

- `distTo[v]` is length of shortest known path from s to v.
- `distTo[w]` is length of shortest known path from s to w.
- `edgeTo[w]` is last edge on shortest known path from s to w.
- If e = v→w gives shorter path to w through v, update both `distTo[w]` and `edgeTo[w]`.

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
   }
}
```


Shortest-paths optimality conditions

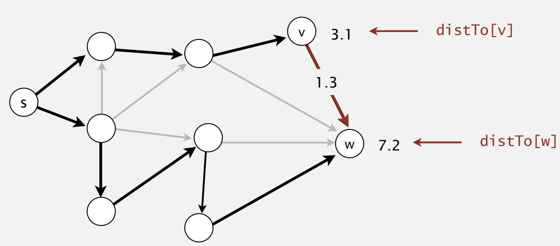
Proposition. Let G be an edge-weighted digraph. Then `distTo[]` are the shortest path distances from s iff:

- For each vertex v, `distTo[v]` is the length of some path from s to v.
- For each edge e = v→w, `distTo[w] ≤ distTo[v] + e.weight()`.

No easy shortcuts exist!

Pf. ⇐ [necessary]

- Suppose that `distTo[w] > distTo[v] + e.weight()` for some edge e = v→w.
- Then, e gives a path from s to w (through v) of length less than `distTo[w]`.



Necessary condition rephrased: If the graph is optimal, there are no easy shortcuts.

Shortest-paths optimality conditions

Proposition. Let G be an edge-weighted digraph. Then `distTo[]` are the shortest path distances from s iff:

- For each vertex v, `distTo[v]` is the length of some path from s to v.
- For each edge e = v→w, `distTo[w] ≤ distTo[v] + e.weight()`.

Pf. ⇒ [sufficient] Sufficient condition rephrased: If there are no easy shortcuts, the graph is optimal.

- Suppose that $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k = w$ is a shortest path from s to w.
- Then,

$$\begin{aligned} \texttt{distTo}[v_1] &\le \texttt{distTo}[v_0] + e_1\texttt{.weight()} \\ \texttt{distTo}[v_2] &\le \texttt{distTo}[v_1] + e_2\texttt{.weight()} \\ &\ldots \\ \texttt{distTo}[v_k] &\le \texttt{distTo}[v_{k-1}] + e_k\texttt{.weight()} \end{aligned}$$

e_i = i^{th} edge on shortest path from s to w

- Add inequalities; simplify; and substitute $\texttt{distTo}[v_0] = \texttt{distTo}[s] = 0$:

$$\texttt{distTo[w]} = \texttt{distTo}[v_k] \le \underbrace{e_1\texttt{.weight()} + e_2\texttt{.weight()} + \ldots + e_k\texttt{.weight()}}_{\text{weight of shortest path from s to w}}$$

- Thus, `distTo[w]` is the weight of shortest path to w. ▪

weight of some path from s to w

Generic shortest-paths algorithm

Generic algorithm (to compute SPT from s)

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat until optimality conditions are satisfied:
- **Relax any edge.**

Optimality conditions:
1. distTo[] is the length of some path (not infinity)
2. No easy shortcuts exist.

Proposition. Generic algorithm computes SPT (if it exists) from `s`.

Pf sketch.
- Throughout algorithm, `distTo[v]` is the length of a simple path from `s` to v (and `edgeTo[v]` is last edge on path).
- Each successful relaxation decreases `distTo[v]` for some `v`.
- The entry `distTo[v]` can decrease at most a finite number of times. ▪

Generic shortest-paths algorithm

Generic algorithm (to compute SPT from s)

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat until optimality conditions are satisfied:
- **Relax any edge.**

Efficient implementations. How to choose which edge to relax?

Ex 1. Dijkstra's algorithm (nonnegative weights).

Ex 2. Topological sort algorithm (no directed cycles).

Ex 3. Bellman-Ford algorithm (no negative cycles).

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4.4 Shortest Paths

- APIs
- shortest-paths properties
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights

Edsger W. Dijkstra: select quotes

" The competent programmer is fully aware of the strictly limited size of his own skull; therefore he approaches the programming task in full humility, and among other things he avoids clever tricks like the plague. "

" In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind. "

" The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence. "

" It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration. "

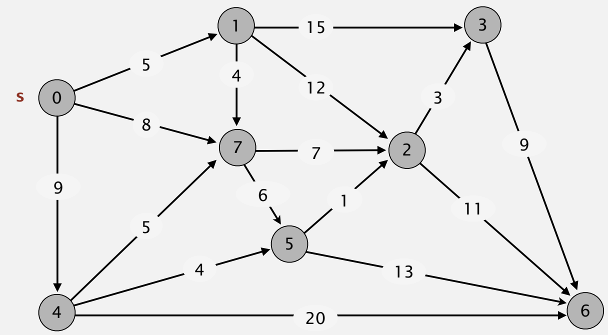
Edsger W. Dijkstra
Turing award 1972

http://www.cs.utexas.edu/users/EWD/transcriptions/

Edsger W. Dijkstra: select quotes

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.



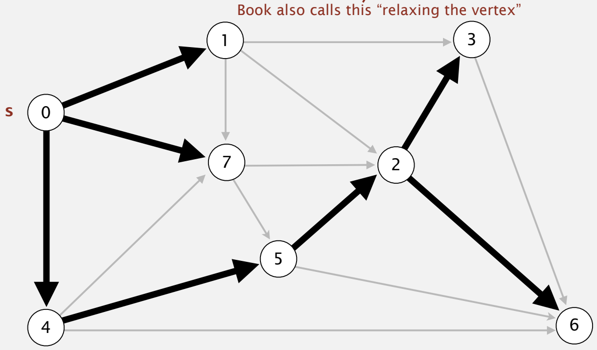
an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

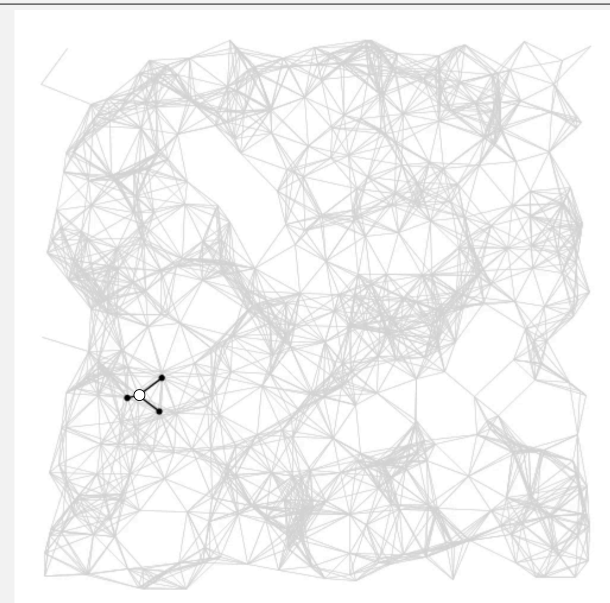
Book also calls this "relaxing the vertex"



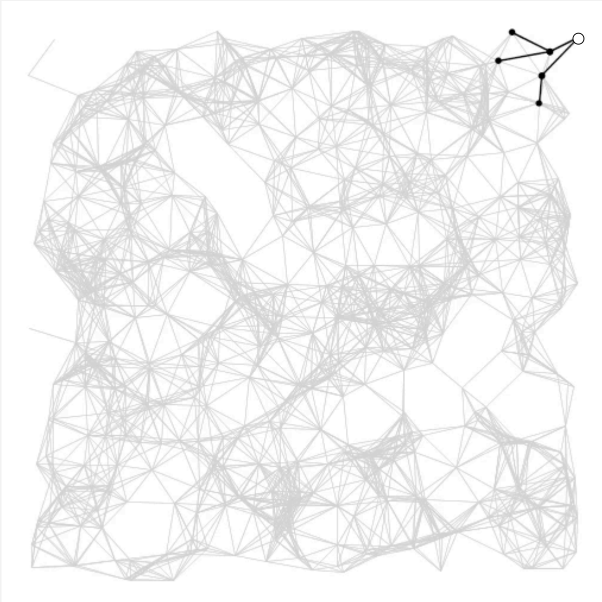
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Dijkstra's algorithm visualization



Dijkstra's algorithm visualization



Dijkstra's algorithm: correctness proof

Proposition. Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

Pf.

- Each edge e = v→w is relaxed exactly once (when v is relaxed), leaving `distTo[w] ≤ distTo[v] + e.weight()`.
- Inequality holds until algorithm terminates because:
  - `distTo[w]` cannot increase ← `distTo[]` values are monotone decreasing
  - `distTo[v]` will not change ← we choose lowest `distTo[]` value at each step (and edge weights are nonnegative)
- Thus, upon termination, shortest-paths optimality conditions hold. ▪

Dijkstra's algorithm: Java implementation

```
public class DijkstraSP
{
   private DirectedEdge[] edgeTo;
   private double[] distTo;
   private IndexMinPQ<Double> pq;

   public DijkstraSP(EdgeWeightedDigraph G, int s)
   {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];
      pq = new IndexMinPQ<Double>(G.V());

      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      pq.insert(s, 0.0);
      while (!pq.isEmpty())
      {
          int v = pq.delMin();
          for (DirectedEdge e : G.adj(v))
             relax(e);
      }
   }
}
```

Essentially the same thing as an ExtrinsicMinPQ

relax vertices in order of distance from s

Dijkstra's algorithm: Java implementation

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
       if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
       else                pq.insert     (w, distTo[w]);
   }
}
```

update PQ

Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: V insert, V delete-min, E decrease-key.

PQ implementation	insert (V inserts)	delete-min (V delete-mins)	decrease-key (E decrease-keys)	total
unordered array	1	V	1	V^2
binary heap	log V	log V	log V	E log V
d-way heap (Johnson 1975)	$d \log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
Fibonacci heap (Fredman-Tarjan 1984)	1 †	log V †	1 †	E + V log V

† amortized

Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

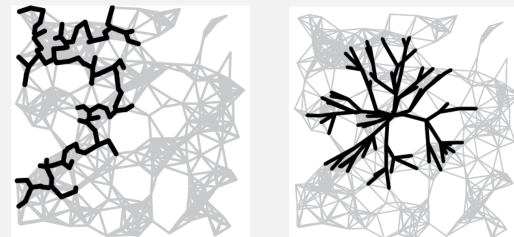
Dijkstra vs. Prim summary

Dijkstra's and Prim's are essentially the same algorithm.

- Both are in a family of algorithms that compute a spanning tree for a graph.

Main distinction: Rule used to choose next vertex for the tree.

- Prim's: Closest vertex to the tree (via an undirected edge).
- Dijkstra's: Closest vertex to the source (via a directed path).



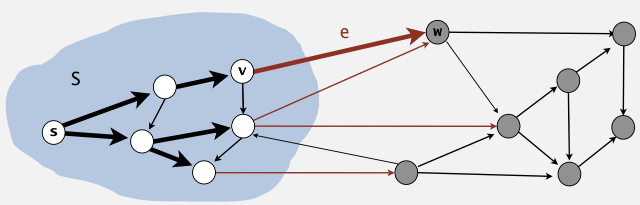
Note: DFS and BFS are also in this family of algorithms.

Priority-first search

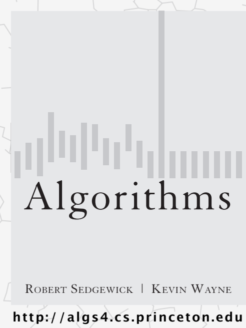
Insight. Four of our graph-search methods are the same algorithm!

- Maintain a set of explored vertices S.
- Grow S by exploring edges with exactly one endpoint leaving S.

DFS. Take edge from vertex which was discovered most recently.
BFS. Take edge from vertex which was discovered least recently.
Prim. Take edge of minimum weight.
Dijkstra. Take edge to vertex that is closest to S.



Challenge. Express this insight in reusable Java code.



4.4 Shortest Paths

- APIs
- shortest-paths properties
- Dijkstra's algorithm
- *edge-weighted DAGs*
- negative weights

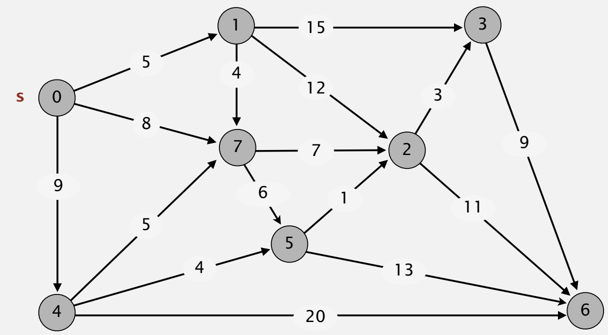
Acyclic edge-weighted digraphs

Q. Suppose that an edge-weighted digraph has no directed cycles. Is it easier to find shortest paths than in a general digraph?

A. Yes!

Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges pointing from that vertex.



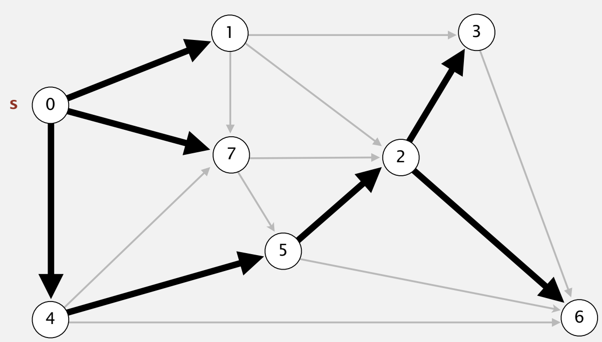
an edge-weighted DAG

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges pointing from that vertex.

0 1 4 7 5 2 3 6



shortest-paths tree from vertex s

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

Shortest paths in edge-weighted DAGs

Proposition. Topological sort algorithm computes SPT in any edge-weighted DAG in time proportional to $E + V$.

edge weights can be negative!

Pf.

- Each edge `e = v→w` is relaxed exactly once (when `v` is relaxed), leaving `distTo[w] ≤ distTo[v] + e.weight()`.
- Inequality holds until algorithm terminates because:
  - `distTo[w]` cannot increase ← `distTo[]` values are monotone decreasing
  - `distTo[v]` will not change ← because of topological order, no edge pointing to v will be relaxed after v is relaxed
- Thus, upon termination, shortest-paths optimality conditions hold. ▪

Shortest paths in edge-weighted DAGs

```
public class AcyclicSP
{
   private DirectedEdge[] edgeTo;
   private double[] distTo;

   public AcyclicSP(EdgeWeightedDigraph G, int s)
   {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];

      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      Topological topological = new Topological(G);
      for (int v : topological.order())
         for (DirectedEdge e : G.adj(v))
            relax(e);
   }
}
```

topological order

Content-aware resizing

Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.

http://www.youtube.com/watch?v=vIFCV2spKtg

Content-aware resizing

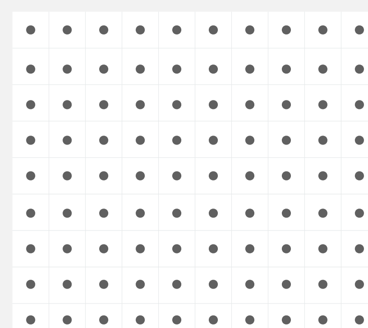
Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.

In the wild. Photoshop CS 5, Imagemagick, GIMP, ...

Content-aware resizing

To find vertical seam:

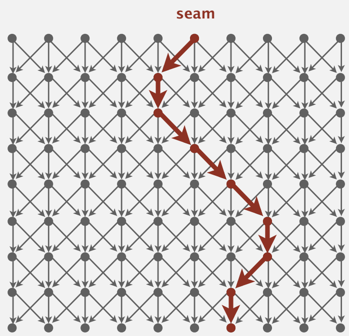
- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path (sum of vertex weights) from top to bottom.



Content-aware resizing

To find vertical seam:

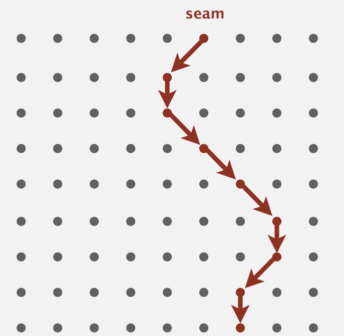
- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path (sum of vertex weights) from top to bottom.



Content-aware resizing

To remove vertical seam:

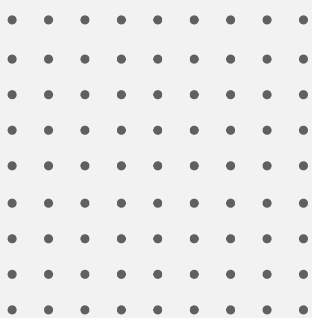
- Delete pixels on seam (one in each row).



Content-aware resizing

To remove vertical seam:

- Delete pixels on seam (one in each row).



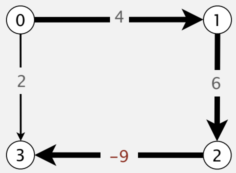
http://algs4.cs.princeton.edu

4.4 Shortest Paths

- APIs
- shortest-paths properties
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights

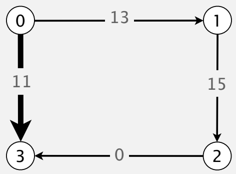
Shortest paths with negative weights: failed attempts

Dijkstra. Doesn't work with negative edge weights.



Dijkstra selects vertex 3 immediately after 0.
But shortest path from 0 to 3 is 0→1→2→3.

Re-weighting. Add a constant to every edge weight doesn't work.



Adding 9 to each edge weight changes the shortest path from 0→1→2→3 to 0→3.

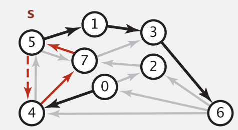
Conclusion. Need a different algorithm.

Negative cycles

Def. A negative cycle is a directed cycle whose sum of edge weights is negative.

digraph

```
4->5  0.35
5->4 -0.66
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```



negative cycle (-0.66 + 0.37 + 0.28)
5->4->7->5

shortest path from 0 to 6
0->4->7->5->4->7->5...->1->3->6

Proposition. A SPT exists iff no negative cycles.

assuming all vertices reachable from s

Bellman-Ford algorithm

Bellman-Ford algorithm

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

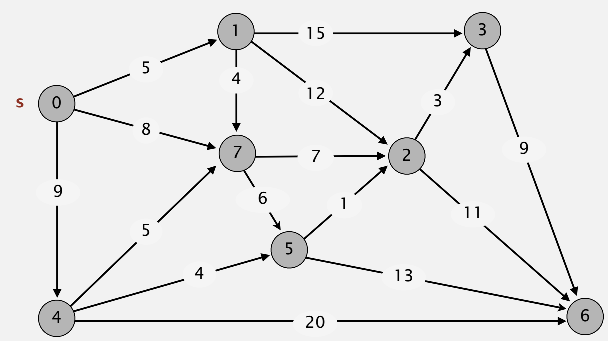
Repeat V times:
- **Relax each edge.**

```
for (int i = 0; i < G.V(); i++)
   for (int v = 0; v < G.V(); v++)
      for (DirectedEdge e : G.adj(v))
         relax(e);
```

pass i (relax each edge)

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.

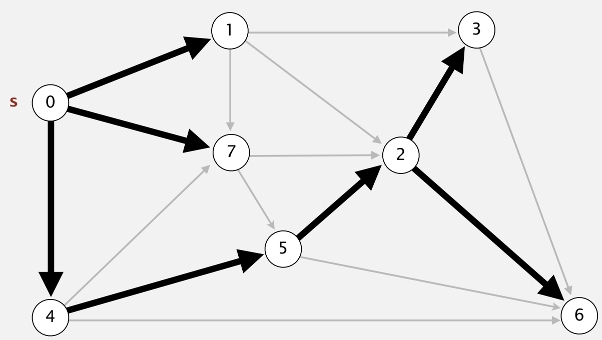


an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Bellman-Ford algorithm demo

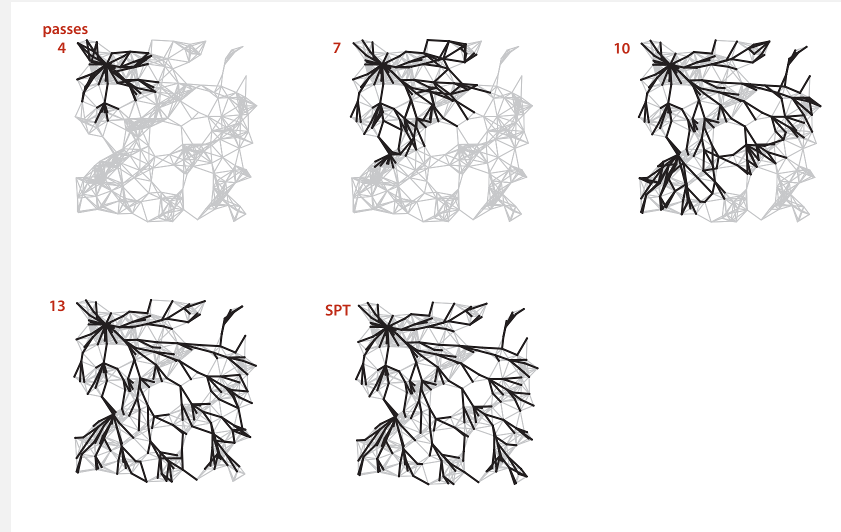
Repeat V times: relax all E edges.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Bellman-Ford algorithm visualization



Bellman-Ford algorithm: analysis

Bellman-Ford algorithm

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat V times:
- **Relax each edge.**

Proposition. Dynamic programming algorithm computes SPT in any edge-weighted digraph with no negative cycles in time proportional to $E \times V$.

Pf idea. After pass i, found shortest path containing at most i edges.

Bellman-Ford algorithm: practical improvement

Observation. If `distTo[v]` does not change during pass `i`, no need to relax any edge pointing from `v` in pass `i+1`.

FIFO implementation. Maintain queue of vertices whose `distTo[]` changed.

be careful to keep at most one copy
of each vertex on queue (why?)

Overall effect.
- The running time is still proportional to $E \times V$ in worst case.
- But much faster than that in practice.

Single source shortest-paths implementation: cost summary

algorithm	restriction	typical case	worst case	extra space
topological sort	no directed cycles	E + V	E + V	V
Dijkstra (binary heap)	no negative weights	E log V	E log V	V
Bellman-Ford	no negative cycles	E V	E V	V
Bellman-Ford (queue-based)		E + V	E V	V

Remark 1. Directed cycles make the problem harder.
Remark 2. Negative weights make the problem harder.
Remark 3. Negative cycles makes the problem intractable.

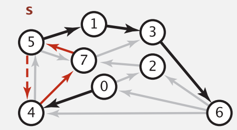
Finding a negative cycle

Negative cycle. Add two method to the API for SP.

```
boolean hasNegativeCycle()                 is there a negative cycle?
Iterable <DirectedEdge> negativeCycle()    negative cycle reachable from s
```

digraph

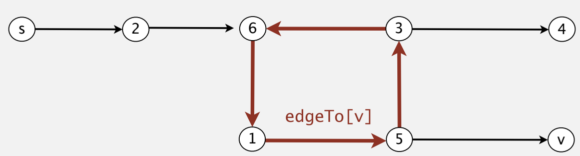
```
4->5  0.35
5->4 -0.66
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```



negative cycle (-0.66 + 0.37 + 0.28)
5->4->7->5

Finding a negative cycle

Observation. If there is a negative cycle, Bellman-Ford gets stuck in loop, updating `distTo[]` and `edgeTo[]` entries of vertices in the cycle.



Proposition. If any vertex v is updated in phase V, there exists a negative cycle (and can trace back `edgeTo[v]` entries to find it).

In practice. Check for negative cycles more frequently.

Negative cycle application: arbitrage detection

Problem. Given table of exchange rates, is there an arbitrage opportunity?

	USD	EUR	GBP	CHF	CAD
USD	1	0.741	0.657	1.061	1.011
EUR	1.350	1	0.888	1.433	1.366
GBP	1.521	1.126	1	1.614	1.538
CHF	0.943	0.698	0.620	1	0.953
CAD	0.995	0.732	0.650	1.049	1

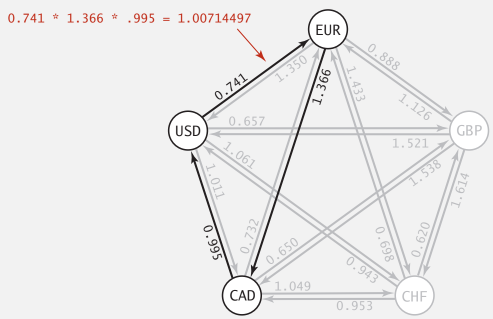
Ex. \$1,000 ⇒ 741 Euros ⇒ 1,012.206 Canadian dollars ⇒ \$1,007.14497.

$1000 \times 0.741 \times 1.366 \times 0.995 = 1007.14497$

Negative cycle application: arbitrage detection

Currency exchange graph.

- Vertex = currency.
- Edge = transaction, with weight equal to exchange rate.
- Find a directed cycle whose product of edge weights is > 1.

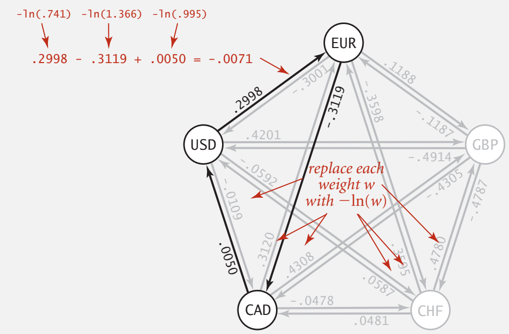


Challenge. Express as a negative cycle detection problem.

Negative cycle application: arbitrage detection

Model as a negative cycle detection problem by taking logs.

- Let weight of edge $v \to w$ be $-\ln$ (exchange rate from currency v to w).
- Multiplication turns to addition; > 1 turns to < 0.
- Find a directed cycle whose sum of edge weights is < 0 (negative cycle).



Remark. Fastest algorithm is extraordinarily valuable!

Shortest paths summary

Dijkstra's algorithm.

- Nearly linear-time when weights are nonnegative.
- Generalization encompasses DFS, BFS, and Prim.

Acyclic edge-weighted digraphs.

- Arise in applications.
- Faster than Dijkstra's algorithm.
- Negative weights are no problem.

Negative weights and negative cycles.

- Arise in applications.
- If no negative cycles, can find shortest paths via Bellman-Ford.
- If negative cycles, can find one via Bellman-Ford.

Shortest-paths is a broadly useful problem-solving model.

BFS code but with Stack – DFS?

No this won't be on the exam...

```
private void mysterySearch(Graph G, Iterable<Integer> sources) {
   Stack<Integer> q = new Stack<Integer>();
   for (int s : sources) {
      q.push(s);
      marked[s] = true;
   }
   while (!q.isEmpty()) {
      int v = q.pop();
      for (int w : G.adj(v)) {
         if (!marked[w]) {
            q.push(w);
            marked[w] = true;
         }
      }
   }
}
```

Problem to be discussed at end of class Tuesday, November 12th

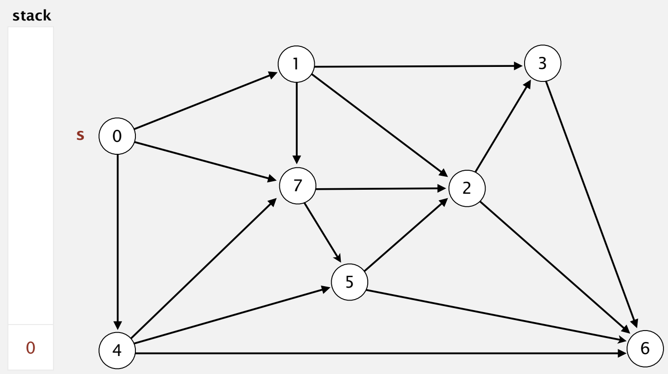
Q: What sort of search does the code above perform?

A. DFS

B. BFS

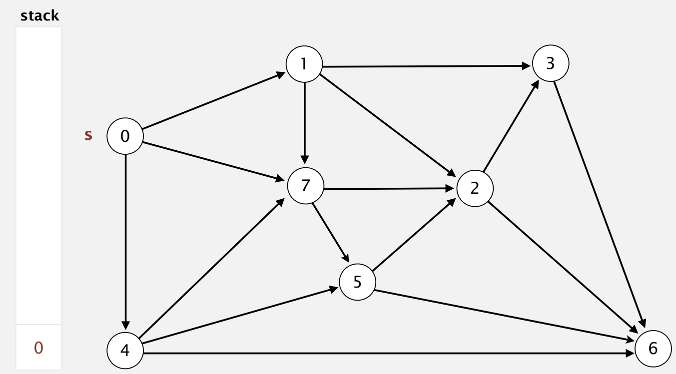
C. Some other type of search

Running BFS code with a stack instead of a queue



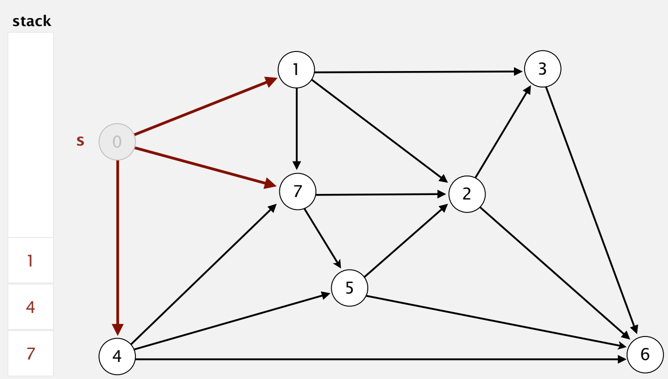
v	distTo[]	edgeTo[]
0	0	–
1	–	–
2	–	–
3	–	–
4	–	–
5	–	–
6	–	–
7	–	–

Running BFS code with a stack instead of a queue



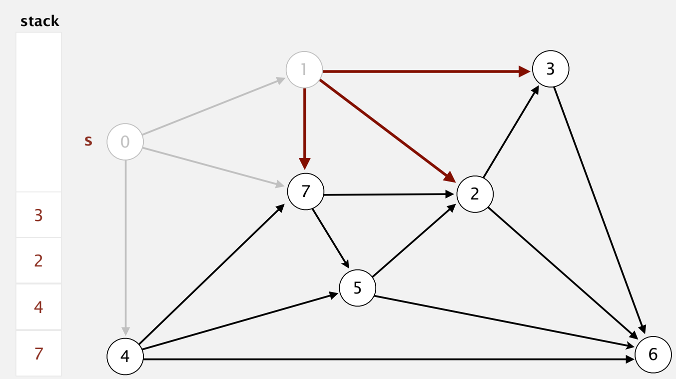
v	distTo[]	edgeTo[]
0	0	–
1	–	–
2	–	–
3	–	–
4	–	–
5	–	–
6	–	–
7	–	–

Running BFS code with a stack instead of a queue



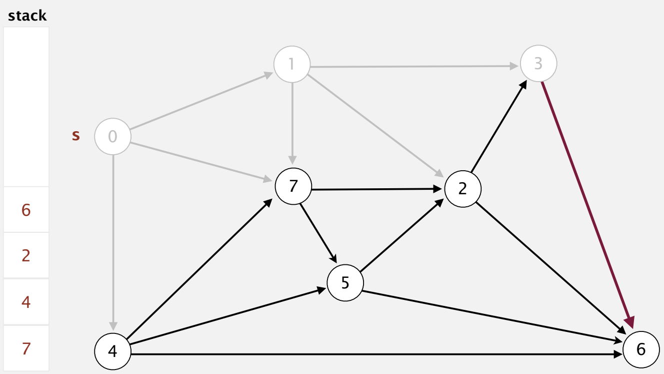
v	distTo[]	edgeTo[]
0	0	–
1	1	0
2	–	–
3	–	–
4	1	0
5	–	–
6	–	–
7	1	0

Running BFS code with a stack instead of a queue



v	distTo[]	edgeTo[]
0	0	–
1	1	0
2	2	1
3	2	1
4	1	0
5	–	–
6	–	–
7	1	0

Running BFS code with a stack instead of a queue



v	distTo[]	edgeTo[]
0	0	-
1	1	0
2	2	1
3	2	1
4	1	0
5	-	-
6	3	3
7	1	0

You get the idea.

Q: What sort of search does the code above perform?

A. DFS

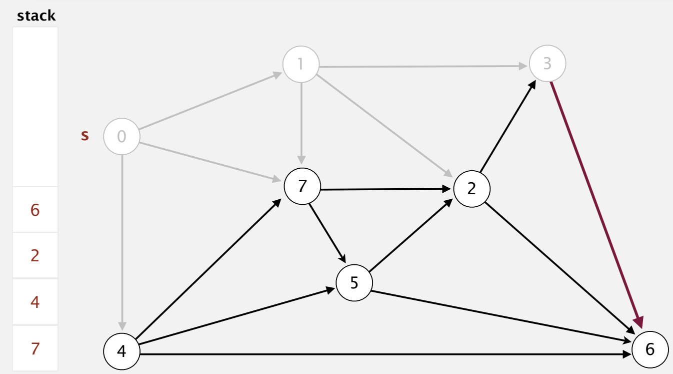
B. BFS

C. Some other type of search - Sort of like a leaky DFS.

Running BFS code with a stack instead of a queue

Q: What sort of search does the code above perform?

C. Some other type of search - Sort of like a leaky DFS.



v	distTo[]	edgeTo[]
0	0	-
1	1	0
2	2	1
3	2	1
4	1	0
5	-	-
6	3	3
7	1	0

Fixing the problem

http://algs4.cs.princeton.edu/41undirected/NonrecursiveDFS.java.html

- Only add one edge to the stack at a time (trickier than you'd think!)
- Use a stack of edges instead of a stack of vertices.
- Use a fancier stack (see the Jiang technique on the booksite for 4.1)